1 Hyperelliptic Surfaces

A compact Riemann surface is called hyperelliptic if it is a double cover of $\mathbb{C}\mathrm{P}^1$, equivalently if there exists a meromorphic function on the surface with exactly two poles counting with multiplicity.

1. Consider a surface M defined by an equation

$$w^2 = a \prod_{i=1}^{d} (z - z_i) \tag{1}$$

in $\mathbb{C}^2$, where a is a non-zero constant and z_i's are distinct points in $\mathbb{C}$. Show that the two projections $(z, w) \mapsto z$ and $(z, w) \mapsto w$ induce charts of M (as a non-compact Riemann surface). Around which points should you use the second projection to define a chart? Show that $M \ni (z, w) \mapsto z \in \mathbb{C}$ is a double cover of $\mathbb{C}$ with branch points $z_1, \cdots z_d$.

2. Now identify $\mathbb{C}$ (of z coordinate) with $\mathbb{C}\mathrm{P}^1 \setminus \{\infty\}$. We will extend the double covering $M \ni (z, w) \mapsto z \in \mathbb{C}$ to $\mathbb{C}\mathrm{P}^1$. For this purpose, let $\tilde{z} := \frac{1}{z}$ be a coordinate around $\infty \in \mathbb{C}\mathrm{P}^1$. By properly changing the coordinate w and using $\tilde{z}$, obtain a regular equation defined around $\tilde{z} = 0$ which coincides with equation 1 over the intersection with $\mathbb{C}$. Show that we obtain a compact Riemann surface $\overline{M}$ and a branched double covering $\pi : \overline{M} \to \mathbb{C}\mathrm{P}^1$ so that $\pi^{-1}(\mathbb{C})$ is the same as M. What is the difference between the cases that d is odd and even? Is it possible to think that the compactification is done in a line bundle of degree k over $\mathbb{C}\mathrm{P}^1$ if $d = 2k$ or $d = 2k - 1$?

3. For $d = 3$, we can homogenize the equation 1 to obtain a compact Riemann surface sitting in $\mathbb{C}\mathrm{P}^2$. Is this surface holomorphically diffeomorphic to the one we obtained in part 2 compactifying M in a line bundle? Does this compactification in $\mathbb{C}\mathrm{P}^2$ work for other values of d? Why or why not?

4. Recall that dz defines a meromorphic section of $K_{\mathbb{C}\mathrm{P}^1}$. The pullback $\pi^*(dz)$ then defines a meromorphic section of $K_{\overline{M}}$. By counting zeros and poles of $\pi^*(dz)$, find the degree of $K_{\overline{M}}$. Identify the genus of $\overline{M}$. See problem 2 in section 2 for the degree of canonical bundles.

This is essentially the proof of Riemann-Hurwitz formula for our special case.

5. Show that any meromorphic function on $\overline{M}$ is expressed as a rational function of z and w. Note that the non-trivial deck transformation σ on $\overline{M}$ as a double cover of $\mathbb{C}\mathrm{P}^1$ is an involution, i.e. $\sigma^2 = \mathrm{id}_{\overline{M}}$, called the hyperelliptic involution. Decompose a given meromorphic function into σ-invariant and σ-anti-invariant part.

6. Show that the space $H^0(\overline{M}, K_{\overline{M}})$ of holomorphic sections of $K_{\overline{M}}$ is a finite dimensional vector space. Find a basis of this space. What is the dimension?

Double-check that the dimension you got is correct by computing the dimension using Riemann-Roch theorem.

2 Degree of Holomorphic Line Bundles

We recall the definition of the degree of line bundles.

Definition 1. *Let L be a (not necessarily holomorphic) line bundle over a compact Riemann surface M and ∇ be any connection on L, then we define the degree of L by*

$$\deg(L) := \frac{i}{2\pi} \int_M F^\nabla \; .$$

Here F^∇ is the curvature 2-form of ∇.

Remark 1. *It is known that the definition does not depend on the choice of ∇. See problem 3 and 4 in section 3*

1. Let TM be the tangent bundle of a compact Riemann surface M which is considered as a holomorphic line bundle. Induce a metric h on TM compatible with the holomorphic structure of M. This means that with respect to holomorphic coordinates, h is locally given as a conformally flat metric. Let ∇ be the Levi-Civita connection for metric h. Show that $F^\nabla = -iK_h \mathrm{vol}_h$, where K_h is the sectional curvature defined by

$$K_h = h(F^\nabla(u,v)v,u)$$

in every local neighborhood with any orthonormal frame (u,v) and vol_h is the volume form.

2. Applying Gauss-Bonnet theorem, show that the degree of the tangent line bundle of a compact Riemann surface equals $2-2g$, thus that the degree of the canonical bundle equals $2g-2$.

For the following problems, use the fact that any holomorphic line bundle has a non-zero meromorphic section and that the degree of the line bundle is given by the total degree of the section. (See Problem 3 and 4 in section 3)

3. Show that any holomorphic line bundle over a compact Riemann surface can be expressed as a tensor product of point bundles and their inverses.

4. For holomorphic line bundles $L_1 = \bigotimes_{i=1}^N L(p_i)^{n_i}$ and $L_2 = \bigotimes_{j=1}^M L(q_j)^{m_j}$ over $\mathbb{C}\mathrm{P}^1$, show that there exists a meromophic function over $\mathbb{C}\mathrm{P}^1$ whose divisor is equal to $\sum_{i=1}^N n_i \cdot p_i - \sum_{j=1}^M m_j \cdot q_j$ if and only if $\deg(L_1) = \deg(L_2)$. Conclude that two holomorphic line bundles over $\mathbb{C}\mathrm{P}^1$ are isomorphic if and only if their degrees are equal. Based on this fact we write $\mathcal{O}(k)$ for the holomorphic line bundle of degree k over $\mathbb{C}\mathrm{P}^1$.

3 Gauge Transformation and Local Expressions of Curvature

Definition 2. *Let E be a vector bundle with a given connection ∇. If a bundle isomorphism $g : E \to E$ is given, then this induces a connection $g \circ \nabla \circ g^{-1}$ on E, which is called the gauge transformation of ∇ through g.*

1. Recall that for a local frame φ_i of vector bundle E, the connection 1-form $\omega_i \in \Omega^1(U_i, \mathfrak{gl}_r(\mathbb{C}))$ associated with ∇ is defined by $\nabla\varphi_i = \varphi_i \cdot \omega_i$. Thus through the local trivializations we get a connection $d + \omega_i$ on each trivial bundle $U_i \times \mathbb{C}^r$. Explicitly express the gauge transformation of $d + \omega_i$ through the transition function g_{ji} (restricted on $U_i \cap U_j$), where $g_{ji} \in C^\infty(U_i \cap U_j, \mathrm{GL}_r(\mathbb{C}))$ is given by $\varphi_i = \varphi_j g_{ji}$. This shows how the local expressions of the connection are transformed.

2. The curvature tensor $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$ is defined as $F^\nabla = (d^\nabla)^2 = d^\nabla \circ \nabla$. Show that the curvature is locally given by $F^\nabla \varphi_i = \varphi_i\,(d\omega_i + \omega_i \wedge \omega_i)$ in terms of the connection 1-form ω_i. What curvature will you get if you gauge transform $d + \omega_i$ by g_{ji}? This is how the local expressions of the curvature are transformed through the transition functions g_{ji}. What if the rank of E equals one?

3. Now let L be a holomorphic *line* bundle over a compact Riemann surface M and $\varphi \in \mathcal{M}(M, L)$ be a meromorphic section. So φ trivializes L away from its zeros and poles. Prove that

$$\frac{i}{2\pi}\int_M F^\nabla = \sum_{z \in M} \mathrm{ord}_z(\varphi)\ .$$

4. Applying Riemann-Roch theorem, we can prove that for every line bundle there exists a meromorphic section (how?). With this fact and the formula from part 3, show that the definition of the degree of a line bundle does not depend on the choice of the connection, that the total order of meromorphic sections of a given bundle does not depend on the choice of meromorphic sections, and that the degrees of line bundles are integers.

4 List of Theorems

Theorem 1. (*Gauss-Bonnet*) *Let M be a smooth surface (mannifold of real dimension 2) of genus g without boundary, h be a Riemannian metric, K_h be the sectional curvature, and vol_h be the volume form. Then it follows*

$$\int_M K_h \mathrm{vol}_h = 4\pi(1-g) \ .$$

Theorem 2. (*Riemann-Hurwitz*) *Let M and N be compact Riemann surfaces and $f : M \to N$ a holomorphic map. Then it follows*

$$2g_M - 2 = \deg(f)\,(2g_N - 2) + \sum_{p\in M} (e_p - 1) \ ,$$

where deg(f) is the degree of f as covering map, e_p is the ramification index at $p \in M$, and g_M and g_N are the genus of M and N respectively.

Theorem 3. (*Riemann-Roch*) *For any holomorphic line bundle L over a compact Riemann surface M, it follows*

$$\dim\left(H^0(M,L)\right) - \dim\left(H^0(M,KL^{-1})\right) = \deg(L) - g + 1 \ \ ,$$

where $H^0(M,L)$ and $H^0(M,KL^{-1})$ denote the spaces of holomorphic sections of bundle L and KL^{-1} respectively.