CORONA LECTURES ON

COMPLEX DIFFERENTIAL GEOMETRY

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Chapter 1

Complex Manifolds

1.1 Real manifolds

This short section is just a reminder of material which should be known. Even if you have not yet seen abstract manifolds, think of submanifolds of a Euclidean space and convince yourself that they satisfy the conditions of the following definition.

Definition 1.1.1. A manifold of dimension n and class C^k, $k \geq 0$, is a Hausdorff topological space M with a countable basis of topology and a covering $\{U_i; i \in I\}$ by open sets such that

(i) each U_i is homeomorphic to an open subset of $\mathbb{R}^n$ via a $\phi : U_i \to \phi(U_i) \subset \mathbb{R}^n$;

(ii) if $U_i \cap U_j \neq \emptyset$, then $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is of class C^k.

The pairs $(U_i, \phi_i)_{i \in I}$ are called *charts*, their collection an *atlas*, and the maps $\phi_i \circ \phi_j^{-1}$ are *transition functions*. A manifold is *smooth* if the transition functions are smooth, and *analytic*, if transition functions are real-analytic.

Smooth functions, smooth maps between manifolds, etc. are defined by passing to the charts. A *tangent vector* v at a point m of a smooth manifold M can be defined either as

(i) an equivalence class of smooth curves $\gamma : (-\epsilon, \epsilon) \to M$, $\gamma(0) = m$, under the relation: $\gamma_1 \sim \gamma_2$ iff $(\phi_i \circ \gamma_1)'(0) = (\phi_i \circ \gamma_2)'(0)$ for some (or any) chart (U_i, ϕ_i) with $m \in U_i$; **or**

(ii) a linear map $L_v : C^\infty(U) \to \mathbb{R}$, U open and containing m, which satisfies the product rule: $L_v(fg) = f(m)L_v(g) + g(m)L_v(f)$.

Remark 1.1.2. Strictly speaking, in (ii) one needs to consider *germs of smooth functions* rather than functions. See any book on differential geometry for the precise definition.

The linear maps L_v are called *derivations* at m. The set of all tangent vectors at m is an n-dimensional vector space called the *tangent space* of M at m, denoted by T_m. The disjoint union $TM = \bigsqcup_{m \in M} T_m M$ has a natural structure of a smooth manifold of dimension $2n$ and the map $\pi : TM \to M$, $\pi(T_m M) = m$, makes it into a vector bundle[1]. Sections of TM, i.e. smooth maps $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$ are called *vector fields*. They can also be defined as *derivations* of the algebra $\mathbb{C}^\infty(M)$, i.e. $\mathbb{R}$-linear maps $L_X : \mathbb{C}^\infty(M) \to \mathbb{C}^\infty(M)$ which satisfy the product rule $L_X(fg) = fL_X(g) + gL_X(f)$.

1.2 Holomorphic Functions

Let V be an n-dimensional complex vector space. Then V can be regarded as a $2n$-dimensional real vector space and the multiplication by i gives a real linear endomorphism

$$J : V \to V \quad \text{with} \quad J^2 = -\mathrm{Id}.$$

If $(a_1, \ldots, a_n)$ is a complex basis of V, then $(a_1, \ldots, a_n, ia_1, \ldots, ia_n)$ is a real basis.
Conversely, given a $2n$-dimensional real vector space V, every real endomorphism $J : V \to V$ with $J^2 = -\mathrm{Id}$ makes V into a complex vector space via

$$(a + ib)v = av + bJ(v), \quad a, b \in \mathbb{R},\ v \in V.$$

Such a J is called a *complex structure*. $-J$ is also a complex structure called the *conjugate complex structure* and $(V, -J)$ is denoted by $\overline{V}$.

Example 1.2.1 (Standard example). $V = \mathbb{C}^n$ with basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$. Then

$$\mathbb{C}^n \simeq \mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in \mathbb{R}\}$$

and the complex structure

$$J(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-y_1, \ldots, -y_n, x_1, \ldots, x_n).$$

We can generalise this example as follows:

Definition 1.2.2. Let E be an n-dimensional real vector space. The *complexification* of E is the real vector space $E^{\mathbb{C}} = E \oplus E$ together with the complex structure

$$J : E^{\mathbb{C}} \to E^{\mathbb{C}}, \quad J(v, w) = (-w, v).$$

$E^{\mathbb{C}}$ is equipped with the *conjugation*

$$C : E^{\mathbb{C}} \to E^{\mathbb{C}}, \quad C(v, w) = (v, -w).$$

Since $C \circ J = -J \circ C$, it is clear that C defines a complex isomorphism between $E^{\mathbb{C}}$ and $\overline{E^{\mathbb{C}}}$.

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[1] If you haven't seen vector bundles yet, don't worry: they'll be discussed later.

Complexification of $\mathbb{R}^n$ is the complex n-space $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$ as above. In this case the conjugation is given by

$$C(z_1, \dots, z_n) = (\overline{z_1}, \dots, \overline{z_n}).$$

If $W = E^{\mathbb{C}} = E \oplus E$ is the complexification of a real vector space E, then the subspace

$$\mathrm{Re}(E) = \{(v, 0) \mid v \in E\}$$

is called the *real part* of W. It is canonically isomorphic to E and we can write $W = E \oplus iE$. An arbitrary complex vector space is the complexification in many different ways (non-canonically): just choose any complex basis B and define E as the real span of B.

Let (V, J) be a real vector space with a complex structure. We complexify V to $V^{\mathbb{C}}$ and extend J (uniquely!) to a complex linear endomorphism of $V^{\mathbb{C}}$:

$$J(v + iw) = J(v) + iJ(w)$$

We still have $J^2 = -\mathrm{Id}$, so the eigenvalues of J are $\pm i$. We set

$$V^{1,0} = \{z \in V^{\mathbb{C}} \mid J(z) = iz\}, \quad V^{0,1} = \{z \in V^{\mathbb{C}} \mid J(z) = -iz\}.$$

These are complex subspaces of $V^{\mathbb{C}}$. Their elements are called vectors of type (1,0) and (0,1) respectively.

Proposition 1.2.3. *The following identities hold:*

(i) $V^{1,0} = \{X - iJX \mid X \in V\}$ *and* $V^{0,1} = \{X + iJX \mid X \in V\}$*;*

(ii) $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ *(as a complex vector space sum);*

(iii) Complex conjugation defines a real linear isomorphism between $V^{1,0}$ *and* $V^{0,1}$*.*

Proof. Obvious. □

Let J be a complex structure on V. Then we obtain a complex structure on V^*:

$$(J\varphi)(v) = \varphi(Jv).$$

Definition 1.2.4. Let (V, J) be a real vector space with a complex structure. A differentiable function

$$f : V \underset{\text{open}}{\supset} U \longrightarrow \mathbb{C} \simeq (\mathbb{R}^2, i)$$

is called *holomorphic* if it's differential commutes with J, i.e.

$$df \circ J = i\, df.$$

Example 1.2.5. Let $V = \mathbb{R}^2$. Then $df|_p$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ which should commute with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A 2×2-matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ commutes with J iff $a_{12} = -a_{21},\ a_{11} = a_{22}$. Thus if $f = u + iv$, then $df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ commutes with J iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the *Cauchy-Riemann equations*. If we introduce differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

then the Cauchy-Riemann equations can be rewritten as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Remark 1.2.6. A holomorphic $f : \mathbb{C}^n \to \mathbb{C}$ can be written locally as a convergent power series in $z_1, \ldots, z_n$ (no $\overline{z_1}, \ldots, \overline{z_n}$ occur).

Observe that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ is a vector of type $(0, 1)$ on $\mathbb{C}^2 = (\mathbb{R}^2)^{\mathbb{C}}$. In general, for an $f : V \to \mathbb{C}$ we can extend $df|_p$ linearly to $V^{\mathbb{C}}$, and then for any $Z = X + iJX \in V^{0,1}$ we have:

$$df|_p(X + iJX) = df|_p(X) + i\, df|_p(JX).$$

This is equal to 0 iff $df|_p(JX) = i\, df|_p(X)$. Thus:

Proposition 1.2.7. *A function $f : (V, J) \longrightarrow \mathbb{C}$ is holomorphic iff*

$$Z(f) = 0 \quad \forall\ Z \in V^{0,1}.$$

□

1.3 Complex manifolds

Definition 1.3.1. A complex manifold of (complex) dimension m is a topological manifold $(M, \mathcal{U})$ (with an atlas $\mathcal{U}$ consisting of charts $\varphi_i : U_i \to \mathbb{C}^m$) such that the transition functions $\varphi_i \circ \varphi_j^{-1}$ are holomorphic maps between open subsets of $\mathbb{C}^m$. In other words we have local complex coordinates on M.

Remark 1.3.2. Obviously a complex manifold of dimension m is smooth (real) manifold of dimension $2m$. We shall denote the underlying real manifold by $M_{\mathbb{R}}$.

Examples 1.3.3. 1) the complex projective space $\mathbb{C}\mathrm{P}^m$ is the set of complex lines in $\mathbb{C}^{m+1}$, i.e.

$$\mathbb{C}\mathrm{P}^m = \mathbb{C}^{m+1}\backslash\{0\}/\sim, \quad \text{where } z \sim w :\Longleftrightarrow \exists \alpha \in \mathbb{C}^* :\ z = \alpha w.$$

Similarly to $\mathbb{R}\mathrm{P}^m$ we define an atlas

$$U_i = \{[z_0, \dots, z_m] \mid z_i \neq 0\}, \quad i = 0, \dots, m,$$
$$\varphi_i : \ U_i \longrightarrow \mathbb{C}^m, \quad [z_0, \dots, z_m] \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_m}{z_i} \right) \in \mathbb{C}^m.$$

The transition functions are

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}(w_1, \dots, w_m) &= \varphi_i([w_1, \dots, w_{j-1}, 1, w_{j+1}, \dots, w_m]) \\ &= \left(\frac{w_1}{w_i}, \dots, \frac{\widehat{w_i}}{w_i}, \dots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \dots, \frac{w_m}{w_i} \right), \end{aligned}$$

hence holomorphic. $\mathbb{C}\mathrm{P}^m$ is compact: We can restrict $\sim$ to the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$

$$S^{2m+1} = \left\{ z_i \in \mathbb{C}^{m+1} \middle| \sum_{i=0}^{m} |z_i|^2 = 1 \right\}.$$

A line $\{\alpha z \mid \alpha \in \mathbb{C}^*\}$ intersects S^{2m+1} in the set $\{\alpha \mid |\alpha|^2 = 1\}$, so in a circle S^1. Hence

$$\mathbb{C}\mathrm{P}^m \simeq S^{2m+1}/S^1$$

as a real manifold (S^1 is viewed as a group acting on S^{2m+1}). E.g. for $m = 1$

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$$

and S^1 acts via $\alpha(z, w) = (\alpha z, \alpha w)$. The quotient is S^2: notice that the following functions on $\mathbb{C}^2$ are invariant under the S^1-action: $a = |z|^2, b = |w|^2$ and $z\overline{w}$ and they satisfy the equation $c\bar{c} = ab$. Hence, if we write $x_1 = \operatorname{Re} c$, $x_2 = \operatorname{Im} c$, $x_3 = |z|^2$, then $x_1^2 + x_2^2 = x_3(1 - x_3)$, which describes a sphere. This projection $S^3 \to S^2$ is called the *Hopf fibration.*

2) More generally, the complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ is the set of all k-dimensional subspaces in $\mathbb{C}^n$. A basis of such a subspace can be written as a $k \times n$-matrix:

$$V = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{k1} & \dots & v_{kn} \end{pmatrix}.$$

Two such matrices define the same subspace if they are transformed into each other by an element $A \in GL(k, \mathbb{C})$ acting by the left multiplication. For each sequence of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $1 \leq \lambda_1 < \dots < \lambda_k \leq n$ we can define a chart U_λ of $\mathrm{Gr}_k(\mathbb{C}^n)$ consisting of subspaces such that the columns with indices λ_i are linearly independent. In other words the minor V_λ consisting of columns with indices λ_i is invertible. The matrix $V_\lambda^{-1} V$ represents the same subspace and its λ_i-th column is e_i. Such a representation is unique. We define

$$\phi_\lambda : U_\lambda \to \mathbb{C}^{k(n-k)}$$

by associating to V the entries of the remaining $n-k$ columns of $V_\lambda^{-1}V$. Check that the transition functions are holomorphic.

Another construction of Grassmannians: $GL(n,\mathbb{C})$ acts transitively on the set of k-dimensional subspaces. The isotropy subgroup of a point, e.g. the subgroup which fixes $S_0 = \langle e_1, \dots, e_k\rangle$ is

$$H = \underbrace{\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)}_{\underbrace{\quad}_{k}\,\underbrace{\quad}_{n-k}} \begin{array}{l} \}k \\ \}n-k \end{array}$$

Thus $\mathrm{Gr}_k(\mathbb{C}^n)$ is the coset space $GL(n,\mathbb{C})/H$. Both $GL(n,\mathbb{C})$ and H are complex Lie groups (open subsets of some $\mathbb{C}^N$) and as for real Lie groups and smooth manifolds one shows that the quotient space (complex Lie group)/(closed complex subgroup) is a complex manifold. As for $\mathbb{CP}^m$, $\mathrm{Gr}_k(\mathbb{C}^n)$ is compact: this time observe that we can choose unitary bases of subspaces, and then $\mathrm{Gr}_k(\mathbb{C}^n) \simeq U(n)/U(k)\times U(n-k)$.

3) As for smooth manifolds, level sets of submersions $f : \mathbb{C}^{m+1} \to \mathbb{C}$ are complex manifolds. If f is holomorphic and the holomorphic differential $df = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{m+1}}\right)$ does not vanish on $f^{-1}(c)$, then $f^{-1}(c)$ is a complex manifold. It is never compact - see homework.
On the other hand, if $p : \mathbb{C}^{m+1} \to \mathbb{C}$ is a homogeneous polynomial, then $v \in p^{-1}(0) \iff \alpha v \in p^{-1}(0)\ \forall \alpha \in \mathbb{C}^*$. Hence, if 0 is the only singular value of p, then we can consider

$$\big(p^{-1}(0)\backslash\{0\}\big)/\!\sim \quad \text{where} \quad v \sim w \Leftrightarrow \exists\ \alpha \in \mathbb{C}^* :\ v = \alpha w,$$

and we obtain a compact complex submanifold of $\mathbb{C}\mathrm{P}^m$.
Important examples of manifolds obtained in this way include the Fermat hypersurfaces $\{[z_0, \dots, z_m] \in \mathbb{C}\mathrm{P}^m \mid z_0^k + \dots + z_m^k = 0\}$.

4) Let D be any lattice in $\mathbb{C}^m$, i.e. a discrete subgroup of the real translation group. Then $\mathbb{C}^m/D$ is a complex manifold, e.g. $\mathbb{C}/(\mathbb{Z}\oplus i\mathbb{Z})$ is the torus.

5) *Hopf manifold*: Let $\lambda > 1$ be a real number. Consider the group $\Gamma \simeq \mathbb{Z}$ of transformations of $\mathbb{C}^m\backslash\{0\}$ given by

$$z \mapsto \lambda^n z, \quad n \in \mathbb{Z}.$$

This is a free and properly discontinuous action and $\mathbb{C}^m\backslash\{0\}/\Gamma$ is a complex manifold. We can identify it as a real manifold. First of all

$$\mathbb{C}^m\backslash\{0\} \simeq \mathbb{R}_{>0}\times S^{2m-1}, \quad z \mapsto \big(\|z\|, z/\|z\|\big).$$

In this representation λ (i.e. $1 \in \mathbb{Z}$) acts by $\lambda.(r,u) = (\lambda r, u)$, and so

$$\mathbb{C}^m\backslash\{0\}/\Gamma \simeq S^1 \times S^{2m-1}.$$

Definition 1.3.4. Let M be a complex manifold. A function $f : M \longrightarrow \mathbb{C}$ is called holomorphic iff for every local holomorphic chart (U, φ) on M, the function $f \circ \varphi^{-1}$ is holomorphic. More generally a map $\varphi : M \longrightarrow M'$ between complex manifolds is called holomorphic iff for every chart (U, φ) on M and (V, ψ) on M', the map $\psi \circ f \circ \varphi^{-1}$ is holomorphic.

We now want to define *holomorphic* tangent vectors. This time the definition in terms of derivations is much more suitable. First of all, for an open subset U of M set:

$$\mathrm{Hol}(U) := \{f : U \longrightarrow \mathbb{C} \mid f \text{ is holomorphic}\}.$$

We now define an *(holomorphic) tangent vector* at $p \in M$ to be a complex derivation of $\mathrm{Hol}(U)$, where U is any connected open neighbourhood of p, i.e. a map $\delta : \mathrm{Hol}(U) \to \mathbb{C}$, such that

$$\begin{aligned} \delta(\alpha f + \beta g) &= \alpha\delta(f) + \beta\delta(g), \quad \forall\, \alpha, \beta \in \mathbb{C}, \\ \delta(fg) &= f(p)\delta(g) + \delta(f)g(p). \end{aligned}$$

This time there is no need for germs, since a holomorphic function on a connected set is determined by its restriction to any open subset. In local complex coordinates $(z_1, \ldots, z_m)$ we can write such a tangent vector v as

$$v = \sum_{i=1}^{m} v_i \frac{\partial}{\partial z_i}.$$

The complex vector space of all holomorphic tangent vectors will be denoted by T_pM (not to be confused with $T_pM_{\mathbb{R}}$).

As for smooth manifolds, we consider the disjoint union

$$TM := \bigsqcup_{p \in M} T_pM.$$

This is again a complex manifold, called the *holomorphic tangent bundle*. The base map is $\pi : TM \to M$, $\pi(T_pM) = p$. A *holomorphic vector field* is a holomorphic map

$$X : M \longrightarrow TM \quad \text{s.t.} \quad \pi \circ X = id|_M.$$

A holomorphic map $F : M \longrightarrow N$ between holomorphic manifolds induces a holomorphic map between tangent bundles

$$F_* : TM \longrightarrow TN, \quad F_*(\delta)(f) = \delta(f \circ F).$$

1.4 Almost complex manifolds

Let M be a complex manifold of real dimension $2n$. Consider $TM_{\mathbb{R}}$ (the real tangent bundle).

Let (U, φ) be a holomorphic chart and define $J : T_pM_{\mathbb{R}} \longrightarrow T_pM_{\mathbb{R}},\ p \in U$ via

$$J(v) = (d\varphi)^{-1} \circ j_n \circ d\varphi(v),$$

where j_n is the standard linear complex structure on $\mathbb{R}^{2n}$

$$j_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-y_1, \ldots, -y_n, x_1, \ldots, x_n).$$

If (V, ψ) is another holomorphic chart around p, then

$$\begin{aligned}(d\psi)^{-1} \circ j_n \circ d\psi(v) &= (d\psi)^{-1} \circ j_n \circ d\underbrace{(\psi \circ \varphi^{-1})}_{\text{holomorphic}} \circ d\varphi(v) \\ &= (d\psi)^{-1} \circ d(\psi \circ \varphi^{-1}) \circ j_n \circ d\varphi(v) = (d\varphi)^{-1} \circ j_n \circ d\varphi(v),\end{aligned}$$

so the definition does not depend on the chart. We obtain an endomorphism of the tangent bundle (i.e. a $(1,1)$-tensor)

$$J : TM_{\mathbb{R}} \longrightarrow TM_{\mathbb{R}}$$

satisfying $J^2 = -\mathrm{Id}$.

Definition 1.4.1. A $(1,1)$-tensor J on a smooth manifold M satisfying $J^2 = -\mathrm{Id}$ is called an *almost complex structure*. The pair (M, J) is then called an *almost complex manifold*.

A complex manifold is therefore canonically an almost complex manifold. We want to investigate the converse. Let (M, J) be an almost complex manifold. Complexify the tangent bundle $T^{\mathbb{C}}M$ (so complexify the vector space T_pM at every point) and consider the subbundles of vectors of type $(1,0)$ and $(0,1)$:

$$\begin{aligned}T^{1,0}M &= \{X - iJX \mid X \in TM\} \ \text{ – the } +i\text{-eigenbundle},\\ T^{0,1}M &= \{X + iJX \mid X \in TM\} \ \text{ – the } -i\text{-eigenbundle}.\end{aligned}$$

Suppose that J arises from complex coordinates $\{z_1, \ldots, z_n\}$ (i.e. (M, J) really is a complex manifold). Then the vectors

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \quad \text{where} \quad \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)$$

are of type $(1,0)$ and

$$\frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}, \quad \text{where} \quad \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$$

of type $(0,1)$. They form bases of $T_p^{1,0}M$ and $T_p^{0,1}M$, respectively. If Z, W are two local sections of $T^{1,0}M$, i.e.

$$Z = \sum_{i=1}^{n} Z_i \frac{\partial}{\partial z_i}, \quad W = \sum_{j=1}^{n} W_i \frac{\partial}{\partial z_j},$$

then

$$[Z, W] = \sum_{i,j=1}^{n} \left(Z_i \frac{\partial W_j}{\partial z_i} - W_i \frac{\partial Z_j}{\partial z_i}\right) \frac{\partial}{\partial z_j}$$

is again a local section of $T^{1,0}M$. Similarly if Z, W are local sections of $T^{0,1}M$, then so is $[Z, W]$. Thus the condition[2] $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$ is a necessary condition for the existence of complex coordinates inducing J. (Formally, this is similar to the *involutivity* required in the Frobenius theorem.)

It turns out that this necessary condition is also sufficient:

Theorem 1.4.2 (Newlander-Nirenberg)**.** *Let (M, J) be an almost complex manifold. The almost complex structure J arises from a holomorphic structure iff*

$$[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M.$$

One says then that J is integrable *and refers to J simply as* complex structure*.*

Let us work out what this condition means. Compute

$$[X + iJX, Y + iJY] = [X, Y] - [JX, JY] + i([JX, Y] + [X, JY]).$$

This should again be of the form $Z + iJZ$, which means that

$$[JX, Y] + [X, JY] = J([X, Y] - [JX, JY]).$$

Equivalently, the tensor[3]

$$N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

vanishes identically. N is called the *Nijenhuis tensor* (or the torsion of an almost complex manifold). Therefore an almost complex structure J arises from complex coordinates (i.e. (M, J)) is a complex manifold) iff the Nijenhuis tensor $N = N_J$ vanishes. The proof of the Newlander-Nirenberg theorem in full generality is much too long to present it here; next week I'll present a proof under the additional assumption that (M, J) is real-analytic. In the meantime, let us look at spheres.

Theorem 1.4.3 (Kirchhoff)**.** *If S^n admits an almost complex structure, then S^{n+1} has trivial tangent bundle.*

Proof. Let J be an almost complex structure on S^n. View S^n as the equator in S^{n+1}, which in turn is the unit sphere in $\mathbb{R}^{n+2}$. Set $e = (0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$, so that every vector $x \in S^{n+1}$ can be written uniquely as $x = ae + by$, $b \geq 0$, $y \in S^n$. Consider

$$T_y S^n = \{z \in \mathbb{R}^{n+1} \mid z \perp y\},$$

and define $\sigma_x : \mathbb{R}^{n+1} \longrightarrow T_x S^{n+1}$ by

$$\begin{aligned} \sigma_x(y) &= ay - be \\ \sigma_x(z) &= az + bJ_y(z), \quad \text{for } \ z \in y^\perp = T_y S^n. \end{aligned}$$

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[2] $[T^{0,1}M, T^{0,1}M]$ is a shorthand for $\left[\Gamma(T^{0,1}M), \Gamma(T^{0,1}M)\right] \subset \Gamma(T^{0,1}M)$.

[3] In Homework 2 you are asked to show that this *is* a tensor.

Let us check that this is in T_xS^{n+1}, i.e. that the right-hand side is orthogonal to $x = ae + by$. Obviously $\langle ay - be, ae + by\rangle = 0$. On the other hand $\sigma_x(z) \perp y$ by definition and, since $\sigma_x(z) \in \mathbb{R}^n$, $\sigma_x(z) \perp e$. Hence $\sigma_x(z) \perp x$. Thus we have a global map $S^n \times \mathbb{R}^{n+1} \longrightarrow TS^{n+1}$, $(x, v) \mapsto \sigma_x(v)$, linear for each x, and we only need to check that it is a bijection for each x. We show that $\mathrm{Ker}(\sigma_x) = 0$. Clearly $\sigma_x(y) \neq 0$. Suppose that $z \neq 0$ and $\sigma_x(z) = 0$. This means that $bJ_y(z) = -az$, and if $b \neq 0$, then z is an eigenvector of J_y with real eigenvalue, which is impossible. On the other hand, if $b = 0$, then $a = 0$, so $x = 0 \notin S^{n+1}$. $\square$

Adams showed in 1960 that TS^{n+1} is trivial if and only if $n + 1 = 1, 3, 7$. Hence only S^2 and S^6 can admit an almost complex structure. For S^2 we already know this, since S^2 is diffeomorphic to $\mathbb{CP}^1$. Here is another description using *quaternions*, i.e. the algebra $\mathbb{H}$ consisting of pairs of complex numbers with coordinate-wise addition and multiplication given by

$$(z_1, z_2)(z_1', z_2') = (z_1 z_1' - z_2 \overline{z'}_2, z_1 z_2' + z_2 \overline{z'}_1).$$

This can be also interpreted by writing an element of $\mathbb{H}$ as $z_1 + z_2 j$, where $j^2 = -1$ and $ij = -ji$. The multiplication is then determined by these identities (plus the associativity and the distributivity). This multiplication is associative, but not commutative.

The quaternionic conjugate of $q = (z_1, z_2)$ is $\overline{q} = (\overline{z}_1, -z_2)$. We have $q\overline{q} = (z_1\overline{z}_1 + z_2\overline{z}_2, 0)$ and we define $|q|^2 = z_1\overline{z}_1 + z_2\overline{z}_2$. A quaternion is called *real* (resp. *purely imaginary*) if $q = \overline{q}$ (resp. $q = -\overline{q}$). q is purely imaginary iff $z_1 = -\overline{z}_1$, so these form a 3-dimensional subspace $\mathrm{Im}\,\mathbb{H}$. The scalar product on $\mathrm{Im}\,\mathbb{H} \simeq \mathbb{R}^3$ is given by $\langle q, q'\rangle = \mathrm{Re}(qq')$ and the vector product by $q \times q' = \mathrm{Im}(qq')$. Now:

$$S^2 = \{q \in \mathrm{Im}\,\mathbb{H} \mid |q| = 1\} \quad \text{and} \quad T_qS^2 = \{q' \in \mathrm{Im}\,\mathbb{H} \mid \langle q, q'\rangle = 0\}.$$

We define $J_q : T_qS^2 \to T_qS^2$ by

$$J_q(q') = q \times q'.$$

Then $J_q^2(q') \in T_qS^2$, since $q \times q' \perp q$. Moreover

$$J_q^2(q') = q \times (q \times q') = q \times (qq' - \mathrm{Re}\, qq') = q \times (qq') = \mathrm{Im}\, q(qq') = \mathrm{Im}\, q^2 q' = -q',$$

since any quaternion in S^2 satisfies $q^2 = -1$. Therefore J is an almost complex structure on S^2.

For S^6 we repeat the procedure. The algebra $\mathbb{O}$ of *Cayley numbers* (or *octonions*) is the set of pairs of quaternions with multiplication

$$(q_1, q_2)(q_1', q_2') = (q_1 q_1' - \overline{q_2'} q_2, q_2' q_2 + q_2 \overline{q_1'}).$$

This multiplication is not even associative. It does, however, satisfy the so-called *alternative law*:

$$x(xx') = (xx)x' \ , \quad (x'x)x = x'(xx),$$

i.e. associativity if two neighbouring factors are the same.

Again we have a conjugation:

$$\overline{(q_1, q_2)} = (\overline{q}_1, -q_2) \quad \text{with} \quad x\overline{x} = (q_1\overline{q}_1 + \overline{q}_2 q_2, 0),$$

and therefore a norm $|x|^2 = q_1\overline{q}_1 + q_2\overline{q}_2$. Again we can define real and purely imaginary Cayley numbers. The vector space of purely imaginary Cayley numbers is 7-dimensional, and it is equipped with a scalar product $\langle x, x'\rangle = -\operatorname{Re}(xx')$ and a vector product $x \times x' = \operatorname{Im}(xx')$. We have $x \times x' = -x' \times x$ and $\langle x \times x', x''\rangle = \langle x, x' \times x''\rangle$. Consider

$$S^6 = \{x \in \operatorname{Im}\mathbb{O} \mid |x| = 1\} \quad \text{and} \quad T_x S^6 = \{y \in \operatorname{Im}\mathbb{O} \mid \langle x, y\rangle = 0\}.$$

Define $J_x(y) = x \times y$. Again $J_x : T_x S^6 \to T_x S^6$ and again $J_x^2 = -\mathrm{Id}$ (observe that in the above calculation of J_q^2 for quaternions one needs exactly the alternative law). This almost complex structure on S^6 has $N \neq 0$, i.e. it is non-integrable. It is unknown whether S^6 admits a complex structure, i.e. whether S^6 is a complex manifold.

We have the following application:

Example 1.4.4. Let M be an oriented hypersurface in $\mathbb{R}^7$. For $m \in M$, consider the unit normal vector ν_m corresponding to the orientation. Then $T_m M \simeq \nu_m^\perp \simeq T_{\nu_m} S^6$. Therefore the almost complex structure on S^6 induces an almost complex structure on M. Thus every oriented hypersurface in $\mathbb{R}^7$ is an almost complex manifold.

1.5 Decomposition of the complexified exterior bundle

Let (M, J) be an almost complex manifold. We have seen that a complex structure on a vector space V induces a complex structure on V^*. Therefore we obtain a complex structure on each $T_m^* M$ and consequently a decomposition of the complexified cotangent bundle

$$(T^*M)^{\mathbb{C}} = T^*M \otimes \mathbb{C}$$

into the $(1,0)$- and $(0,1)$-parts. For convenience, we shall write $\Lambda^1_{\mathbb{C}} = (T^*M)^{\mathbb{C}}$, $\Lambda^{1,0}M = \big((T^*M)^{\mathbb{C}}\big)^{(1,0)}$, and $\Lambda^{0,1}M = \big((T^*M)^{\mathbb{C}}\big)^{(0,1)}$. We have (see §1.2):

$$\Lambda^{1,0}M = \{\varphi - i\varphi \circ J \mid \varphi \in T^*M\}$$
$$\Lambda^{0,1}M = \{\varphi + i\varphi \circ J \mid \varphi \in T^*M\}.$$

Example 1.5.1. On $\mathbb{C}^n$ we have $(Jdx_i)(\frac{\partial}{\partial y_i}) = dx_i(J\frac{\partial}{\partial y_i}) = dx_i(-\frac{\partial}{\partial x_i}) = -1$, so $\Lambda^{1,0} = \{dx_i + idy_i\}$ and $Jdx_i = -dy_i$.

Lemma 1.5.2. *We have*

$$\Lambda^{1,0}M = \{\omega \in \Lambda^1_{\mathbb{C}}M \mid \omega(Z) = 0 \ \forall\ Z \in T^{0,1}M\}$$
$$\Lambda^{0,1}M = \{\omega \in \Lambda^1_{\mathbb{C}}M \mid \omega(Z) = 0 \ \forall\ Z \in T^{1,0}M\}$$

Proof. $\omega \in \Lambda^{1,0}M \iff \omega \circ J = i\omega \iff (\omega \circ J)(V) = i\omega(V)\ \forall\ V \in T^{\mathbb{C}}M$. If we decompose $V = V^{1,0} + V^{0,1}$, then

$$(\omega \circ J)(V) = \omega(JV) = \omega(iV^{1,0} - iV^{0,1}) = i\omega(V^{1,0}) - i\omega(V^{0,1})$$

This is equal to $i\omega(V)$ iff $\omega(V^{0,1}) = 0$. Analogously for $\Lambda^{0,1}M$. □

We now decompose the k-th exterior power $\Lambda^k_{\mathbb{C}}M$ of $T^*M \otimes \mathbb{C}$:

$$\Lambda^k_{\mathbb{C}}M = \Lambda^k(\Lambda^{1,0}M \oplus \Lambda^{0,1}M) = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M).$$

We shall write $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$. If $\varphi_1, \ldots, \varphi_n$ is a basis of $\Lambda^{1,0}_m M$, then $\overline{\varphi}_1, \ldots, \overline{\varphi}_n$ is a basis of $\Lambda^{0,1}_m M$, and the set of alternating forms

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \overline{\varphi}_{j_1} \wedge \cdots \wedge \overline{\varphi}_{j_q}, \quad \text{with } i_1 < \cdots < i_p \leq n,\ j_1 < \cdots < j_q \leq n,$$

is a basis of $\Lambda^{p,q}_m M$. Therefore the rank of $\Lambda^{p,q}M$ is $\binom{n}{p}\binom{n}{q}$.
Sections of $\Lambda^k_{\mathbb{C}}M$ are $\mathbb{C}$-valued differential forms; sections of $\Lambda^{p,q}M$ are called *differential forms of type (or degree) (p,q)* and their space is denoted by $\Omega^{p,q}(M)$.

Proposition 1.5.3.

$$d\Omega^{p,q} \subset \Omega^{p+2,q-1} \oplus \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p-1,q+2}.$$

Proof. Let $\omega \in \Omega^{p,q}(M)$. We can write it locally as

$$\omega = f\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \overline{\varphi}_{j_1} \wedge \cdots \wedge \overline{\varphi}_{j_q},$$

where $\varphi_1, \ldots, \varphi_n$ is a local frame of $(1,0)$-forms. We know that $df \in \Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ and $d\varphi_s \in \Omega^2(M) = \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$ and similarly for $\overline{\varphi}_s$. Applying d to ω decomposed as above proves the claim. □

For integrable almost complex structures this becomes much simpler, since we can choose a frame of the form $\varphi_i = dz_i$, where the z_i are local complex coordinates. Then $d(dz_i) = d(d\bar{z}_i) = 0$, and so

$$\begin{aligned} &d(f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}) \\ &= df \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \in \Omega^{p+1,q} \oplus \Omega^{p,q+1}, \end{aligned}$$

and also $df \in \Omega^{1,0} \oplus \Omega^{0,1}$. In fact we have:

Proposition 1.5.4. *For an almost complex manifold M, the following conditions are equivalent:*

a) If Z and W are complex vector fields of type $(1,0)$, then so is $[Z,W]$.

b) If Z and W are complex vector fields of type $(0,1)$, then so is $[Z,W]$.

c) $d\Omega^{1,0} \subset \Omega^{2,0} \oplus \Omega^{1,1}$ and $d\Omega^{0,1} \subset \Omega^{1,1} \oplus \Omega^{0,2}$.

d) $d\Omega^{p,q} \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1} \quad \forall p, q$.

e) the almost complex structure is integrable (i.e. $N = 0$*).*

Proof. Owing to the Newlander-Nirenberg theorem, we already know that a) $\iff$ b) $\iff$ e). Clearly d) $\implies$ c) and the argument in the proof of Proposition 1.5.3 implies that c) $\implies$ d). It remains to show that c) is equivalent to a) and b). Let ω be a 1-form of type $(0,1)$ and Z, W vector fields of type $(1,0)$. A well-known formula for the exterior derivative gives then

$$d\omega(Z,W) = Z(\underbrace{\omega(W)}_{=0}) - W(\underbrace{\omega(Z)}_{=0}) - \omega([Z,W]) = -\omega([Z,W]). \tag{1.5.1}$$

Observe that the 2nd formula in c) (denote it by c2)) is equivalent to $d\omega(Z,W) = 0$ for all $\omega \in \Omega^{0,1}$ and $(1,0)$ vector fields Z, W. Formula (1.5.1) implies that this is equivalent to $[Z,W]$ being of type $(1,0)$. Thus c2) $\iff$ a). Similarly c1) $\iff$ b). ☐

Given two manifolds M and M' and a smooth map $f : M \longrightarrow M'$, we can extend the differential f_* to a $\mathbb{C}$-linear mapping of $T^{\mathbb{C}}M$ to $T^{\mathbb{C}}M'$, which we still denote by f_*. Similarly[4] f^* maps complex differential forms on M' to complex differential forms on M.

Definition 1.5.5. A smooth map $f : (M,J) \longrightarrow (M',J')$ between almost complex manifolds is called *almost complex* if $f_* \circ J = J' \circ f_*$.

Note that for complex manifolds "almost complex map" is the same as "holomorphic map".

Proposition 1.5.6. *For a smooth map* $f : (M,J) \longrightarrow (M',J')$ *between almost complex manifolds the following conditions are equivalent:*

a) If Z *is a complex tangent vector of type* $(1,0)$ *on* M*, then so is* $f_*(Z)$ *on* M'*.*

b) If Z *is a complex tangent vector of type* $(0,1)$ *on* M*, then so is* $f_*(Z)$ *on* M'*.*

c) If ω *is a complex differential form of type* (p,q) *on* M'*, then* $f^*\omega$ *is a differential form of type* (p,q) *on* M*, for all* p,q*.*

d) f *is almost complex.*

Proof. Homework. ☐

Definition 1.5.7. An *infinitesimal automorphism* of an almost complex structure J on M is a vector field X such that $L_X J = 0$. (In other words, the local flow of X consists of (local) almost complex transformations.)

Proposition 1.5.8. *A vector field* X *is an infinitesimal automorphism of an almost complex structure* J *iff*

$$[X, JY] = J([X,Y]) \quad \forall\ Y \in \Gamma(TM).$$

---

[4] Recall that the *pullback* $f^*\omega$ of a differential k-form is defined by $f^*\omega(X_1, \ldots, X_k) = \omega(f_*X_1, \ldots, f_*X_k)$.

Proof.

$$[X, JY] = L_X(JY) = (L_X J)Y + JL_X Y = (L_X J)Y + J([X, Y]).$$

□

Remark 1.5.9. If X is an infinitesimal automorphism of J, JX need not to be. In fact, the last proposition implies that if X is an infintesimal automorphism, then, for all vector fields Y,

$$N(X, Y) = [JX, JY] - J[JX, Y] - [X, Y] - J[X, JY] = [JX, JY] - J[JX, Y],$$

and so JX is also an infinitesimal automorphism iff $N(X, Y) = 0 \;\forall\; Y$. Conversely, it follows that if $N \equiv 0$, i.e. the almost complex structure J is integrable, then the Lie algebra $\mathfrak{a}$ of infinitesimal automorphisms of J is stable under J, and $[X, JY] = J[X, Y] \;\forall X, Y \;\in \mathfrak{a}$. Hence $\mathfrak{a}$ is a complex Lie algebra (possibly infinite-dimensional).

Proposition 1.5.10. *On a complex manifold M, the Lie algebra of infinitesimal automorphisms of the complex structure J is isomorphic to the Lie algebra of holomorphic vector fields, the isomorphism being given by*

$$X \mapsto Z = \frac{1}{2}(X - iJX).$$

Proof. Suppose that $X - iJX$ is holomorphic and $Y \in \Gamma(TM)$ is arbitrary. If f is a local holomorphic function, then

$$(X + iJX)(f) = 0 \implies (X - iJX)(f) = (2X - (X + iJX))(f) = 2X(f).$$

Hence $X(f)$ is holomorphic, which means that $(Y + iJY)(X(f)) = 0$ and of course $(Y + iJY)(f) = 0$. Therefore

$$[Y + iJY, X](f) = (Y + iJY)(X(f)) - X((Y + iJY)(f)) = 0.$$

On the other hand:

$$\begin{aligned}[Y + iJY, X](f) = 0 &\iff [Y + iJY, X] \;\text{ is of type (0,1)}\\ &\iff \mathrm{Im}([Y + iJY, X]) = J\,\mathrm{Re}([Y + iJY, X])\\ &\iff [JY, X] = J[Y, X] \iff X \in \mathfrak{a}.\end{aligned}$$

Conversely, suppose that X is an infinitesimal automorphism of J. Due to Proposition 1.5.8, we know that $[JY, X] = J[Y, X]$, i.e. $[Y + iJY, X]$ is of type $(0, 1)$ for any vector field Y. Then (reversing the argument above) $[Y + iJY, X](f)$ for any local holomorphic function f, so $(Y+iJY)(X(f)) = 0$, which means that $X(f)$ is holomorphic, and hence $X - iJX$ is holomorphic.

Thus the map $\theta : X \mapsto \frac{1}{2}(X - iJX)$ is a linear isomorphism between infinitesimal automorphism of J and holomorphic vector fields on M. We need to check that this map is a Lie algebra homomorphism:

$$[\theta(X),\theta(Y)] = \frac{1}{4}[X - iJX, Y - iJY] = \frac{1}{2}\left([X,Y] - [JX,JY] - i[JX,Y] - i[X,JY]\right)$$
$$= \frac{1}{4}\left([X,Y] + [X,Y] - iJ[X,Y] - iJ[X,Y]\right) = \frac{1}{2}\left([X,Y] - iJ[X,Y]\right) = \theta([X,Y])$$

□

Definition 1.5.11. A real vector field X on a complex manifold is called *real-holomorphic* if $X - iJX$ is a holomorphic vector field.

We shall now prove the Newlander-Nirenberg theorem in the case when both the manifold and the almost complex structure are real-analytic.

Theorem 1.5.12 (Newlander-Nirenberg theorem in analytic case)**.** *A real analytic almost complex structure with vanishing torsion is integrable, i.e. it is a complex structure.*

Before the formal proof, let me describe the intuitive idea behind it. Since M is real analytic (i.e. the transition functions are), we can *complexify* M, i.e. construct a complex manifold $M^{\mathbb{C}}$ such that M sits inside $M^{\mathbb{C}}$ (as a fixed-point set of an anti-holomorphic involution, but we shall not need this explicitly). In order to do this, extend each transition function $\phi_j \circ \phi_i^{-1} : \phi_i(U_i) \to \phi_j(U_i)$ into the complex domain, i.e. to a small neighbourhood of $\phi_i(U_i)$ in $\mathbb{C}^n$ (small enough so that the extended map remains a diffeomorphism). We can do this by expanding a transition function locally into power series and replacing each real coordinate x_i with a complex coordinate z_i. Since J is real analytic, it extends analogously to a holomorphic endomorphism $J : TM^{\mathbb{C}} \to TM^{\mathbb{C}}$ satisfying $J^2 = -\mathrm{Id}$, where $TM^{\mathbb{C}}$ is the holomorphic vector bundle. We consider the $\pm i$-eigenbundles, denoted by T^+ and T^-. These are complex subbundles of $TM^{\mathbb{C}}$ and they satisfy $[T^{\pm}, T^{\pm}] \subset T^{\pm}$, since these conditions hold on M. Using the holomorphic version of the Frobenius theorem, $M^{\mathbb{C}}$ is foliated into submanifolds, the tangent space of which at each point is T^-. The leaf space (which is well defined at least in a small neighbourhood of each point) is then a complex manifold. In a neighbourhood of each $m \in M$ the leaf space is simply M, and so we obtain local complex coordinates on M. These induce the given J, since J is i on T^+.

We proceed with a more formal proof. We need the following lemma:

Lemma 1.5.13. *Let (M,J) be an almost complex manifold with* $\dim_{\mathbb{R}} M = 2n$*. If every point of M has a neighbourhood U and n complex valued smooth functions $f_1, \ldots, f_n : U \to \mathbb{C}$ such that $df_1, \ldots, df_n$ are of type* $(1,0)$ *and linearly independent at every point of U, then the almost complex structure J is integrable.*

Proof. By taking U small enough, we may assume that $f = (f_1, \ldots, f_n)$ is a diffeomorphism of U onto an open subset of $\mathbb{C}^n$. Let V be a small neighbourhood of another point with a similar $g = (g_1, \ldots, g_n) : V \to \mathbb{C}^n$ (also a diffeomorphism onto image). Suppose that $U \cap V \neq \emptyset$. It follows from Proposition 1.5.6 that f and g are almost complex mappings (since f_* and g_* map $(1,0)$-vectors to $(1,0)$-vectors). Hence $f \circ g^{-1}$ is also almost complex, which is means $f \circ g^{-1}$ is holomorphic, since the almost complex structure of $\mathbb{C}^n$ is integrable. Thus we obtain a complex atlas on M, which induces the given almost complex structure J. □

Proof of the Newlander-Nirenberg theorem. An immediate consequence of the above lemma is that we only need to prove the theorem locally. Let U be a small neighbourhood in M and $x^1, \ldots, x^{2n}$ (real-analytic) local coordinates in U. The 1-forms

$$dx^1 + iJdx^1, \ldots, dx^{2n} + iJdx^{2n}$$

span $\Lambda^{1,0}_x M$ at each x, so we can choose n among them which are linearly independent everywhere on U (perhaps after making U smaller). Denote these by $\omega^1, \ldots, \omega^n$ and write

$$\omega^i = \sum_{\alpha=1}^{2n} f^i_\alpha(x) dx^\alpha.$$

The assumption that J is real analytic means that the coefficients $f^i_\alpha(x)$ are real analytic (and $\mathbb{C}$-valued). We may consider U as a neighborhood of the origin in $\mathbb{R}^{2n}$ with coordinates $x^1, \ldots, x^{2n}$. Complexify $\mathbb{R}^{2n}$ to $\mathbb{C}^{2n}$ with coordinates $z^1, \ldots, z^{2n}$ where $z^i = x^i + \sqrt{-1}\, y^i$.

Since $f^i_\alpha(x)$ are real-analytic, we can extend them to holomorphic functions $f^i_\alpha(z)$ on a neighborhood $\widetilde{U}$ of U in $\mathbb{C}^{2n}$ by taking the power series expansion of $f^i_\alpha(x)$ and replacing x with z. Similarly we can extend the complex conjugate functions $\overline{f^i_\alpha}(x)$ to holomorphic functions $\widetilde{f^i_\alpha}(z)$ on $\widetilde{U}$ (maybe after making $\widetilde{U}$ smaller). Set

$$\Omega^i = \sum_{\alpha=1}^{2n} f^i_\alpha(z) dz^\alpha \qquad \text{and} \qquad \widetilde{\Omega}^i = \sum_{\alpha=1}^{2n} \widetilde{f^i_\alpha}(z) dz^\alpha.$$

Since ω^i are linearly independent, so are $\omega^1, \ldots, \omega^n, \overline{\omega}^1, \ldots, \overline{\omega}^n$, i.e. the matrix

$$\begin{bmatrix} f^i_\alpha(x) \\ \widetilde{f^i_\alpha}(x) \end{bmatrix}$$

is nonsingular. Hence the $(2n \times 2n)$-matrix formed by $f^i_\alpha(z)$ and $\widetilde{f^i_\alpha}(z)$ remains nonsingular for z in a small neighborhood of U in $\mathbb{C}^{2n}$ and, consequently, we can take $\widetilde{U}$ small enough so that $\Omega^1, \ldots, \Omega^n, \widetilde{\Omega}^1, \ldots, \widetilde{\Omega}^n$ are linearly independent at each point of $\widetilde{U}$. Therefore we can express each $d\Omega^j$ as

$$d\Omega^j = \sum_{k<l} A^j_{kl}\, \Omega^k \wedge \Omega^l + \sum_{k,l} B^j_{kl}\, \Omega^k \wedge \widetilde{\Omega}^l + \sum_{k<l} C^j_{kl}\, \widetilde{\Omega}^k \wedge \widetilde{\Omega}^l, \tag{1.5.2}$$

where the coefficients are holomorphic functions on $\widetilde{U}$.

On the other hand, the equivalence of conditions a) and c) in Proposition 1.5.4 (which we proved directly, without resorting to the Newlander-Nirenberg theorem) means that $d\omega^j$ is a sum of terms of type $(2,0)$ and $(1,1)$. If we restrict (1.5.2) to U, i.e. to $y = 0$, then it follows that $C^j_{kl}|_U = 0$. Since the functions C^j_{kl} are holomorphic, they must= vanish identically on $\widetilde{U}$. Hence

$$d\Omega^j = \sum_{k<l} A^j_{kl}\ \Omega^k \wedge \Omega^l + \sum_{k,l} B^j_{kl}\ \Omega^k \wedge \widetilde{\Omega}^l.$$

We now appeal to the following holomorphic version of the Frobenius theorem.

Theorem 1.5.14 (Frobenius). *Let* $\varphi^1 \dots, \varphi^r$ *be everywhere linearly independent holomorphic* 1*-forms in a neighborhood* V *of* 0 *in* $\mathbb{C}^m$*. If*

$$d\varphi^j = \sum_{k=1}^{r} \psi^j_k \wedge \varphi^k, \quad j = 1, \dots, r,$$

where each ψ^j_k *is a holomorphic* 1*-form on* V*, then there exists a smaller neighborhood* W *of* 0 *and holomorphic functions* $g^1, \dots, g^r$ *on* W*, such that*

$$\varphi^j = \sum_{k=1}^{r} p^j_k\ dg^k, \quad j = 1, \dots, r,$$

where the p^j_k *are holomorphic functions on* W*.*

Continuation of the proof of the Newlander-Nirenberg theorem. It follows that there exist holomorphic functions $G^1, \dots, G^n$ in a neighborhood of 0 in $\mathbb{C}^{2n}$, such that

$$\Omega^j = \sum_{k=1}^{n} P^j_k dG^k, \quad j = 1, \dots, r.$$

If we write $g^k = G^k|_U$ and $p^j_k = P^j_k|_U$, then $\omega^i = \sum\limits_{k=1}^{n} p^j_k dg^k$.

Since $\omega^1, \dots, \omega^n$ are linearly independent $(1,0)$-forms on U, $dg^1, \dots, dg^k$ are also everywhere linearly independent 1-forms of type $(1,0)$, and the assumption of Lemma 1.5.13 is satisfied. Therefore J is integrable. □

Further reading:

i) A relatively simple proof of the full version of the Newlander-Nirenberg theorem may be found in: L. Nirenberg, “Lectures on Linear Partial Differential Equations”, AMS, 1973.

ii) Whenever you have a geometric structure, you may ask about homogeneous examples and their classification. You can read about invariant almost complex structures on homogeneous manifolds in §X.6 of Kobayashi & Nomizu, vol. II.

iii) In the last question session, you asked about the space of all almost complex structures on a given manifold. The only nontrivial results I found are in the 2018 Ph.D. thesis by Bora Ferlengez "Studying the Space of Almost Complex Structures on a Manifold Using de Rham Homotopy Theory". It is available online at: `https://academicworks.cuny.edu/cgi/viewcontent.cgi?article=3931&context=gc_etds` Be warned, however: this is not easy stuff and will require serious background reading in topology.

1.6 Dolbeault cohomology

Let M be an complex manifold. According to Proposition 1.5.4 we have

$$d\Omega^{p,q}(M) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M).$$

This means that we can decompose the exterior derivative

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

as $d = \partial + \bar{\partial}$, where

$$\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M) \quad \text{and} \quad \bar{\partial} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M).$$

In local coordinates, if we write $\varphi = \sum_I \varphi_I \; dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$ (where I denotes multi-indices), then

$$\bar{\partial}\varphi = \sum_I \sum_s \frac{\partial \varphi_I}{\partial \bar{z}_s} \; d\bar{z}_s \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q},$$

and similarly for ∂.

Lemma 1.6.1. *The following identities hold:*

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proof. We have

$$0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2.$$

Now just observe that, on $\Omega^{p,q}(M)$, ∂^2 takes values in $\Omega^{p+2,q}(M)$, $(\partial\bar{\partial} + \bar{\partial}\partial)$ in $\Omega^{p+1,q+1}(M)$, and $\bar{\partial}^2$ in $\Omega^{p,q+2}(M)$. □

Remark 1.6.2. In local coordinates, the equation $\bar{\partial}^2 = 0$ is equivalent to

$$\frac{\partial^2}{\partial \bar{z}_i \partial \bar{z}_j} = \frac{\partial^2}{\partial \bar{z}_j \partial \bar{z}_i},$$

analogously to d^2.

We denote by $Z^{p,q}_{\bar\partial}(M)$ the space of *$\bar\partial$-closed* forms of type (p,q), i.e.

$$Z^{p,q}_{\bar\partial}(M) = \mathrm{Ker}\left(\bar\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)\right).$$

Since $\bar\partial^2 = 0$, $\mathrm{Im}\,\bar\partial \subset \mathrm{Ker}\,\bar\partial$, and we define the *Dolbeault cohomology groups* of M to be

$$H^{p,q}_{\bar\partial}(M) = \frac{\mathrm{Ker}(\bar\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1})}{\mathrm{Im}(\bar\partial : \Omega^{p,q-1} \longrightarrow \Omega^{p,q}(M))} = \frac{Z^{p,q}_{\bar\partial}(M)}{\bar\partial(\Omega^{p,q-1}(M))}.$$

These are complex vector spaces. Observe (c.f. Prop. 1.5.6) that for a holomorphic map $f : M \to N$ of complex manifolds we have $f^*(\Omega^{p,q}(N)) \subset \Omega^{p,q}(M)$. Moreover $\bar\partial \circ f^* = f^* \circ \bar\partial$, and hence f induces a linear map

$$f^* : H^{p,q}_{\bar\partial}(N) \to H^{p,q}_{\bar\partial}(M).$$

Recall[5] that the key fact about the de Rham cohomology is the *Poincaré Lemma*:

“An open ball in $\mathbb{R}^n$ has trivial de Rham cohomology.”

We have a Dolbeault analogue of this:

Proposition 1.6.3 ($\bar\partial$-Poincaré Lemma)**.** *Let $\Delta = \Delta(r)$ be an open polydisk in $\mathbb{C}^n$, i.e.*

$$\Delta = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_i| < r_i\}, \quad r_i \in (0,\infty].$$

Then $H^{p,q}_{\bar\partial}(\Delta) = 0$ for all $q \geq 1$ and all p.

Remark 1.6.4. On the other hand, observe that $H^{p,0}_{\bar\partial}(\Delta)$ is the infinite-dimensional vector space of holomorphic p-forms on Δ.

Proof. We first consider the case $n = 1$. Observe that if $\dim M = 1$, then $\Omega^{2,0}(M) = \Omega^{0,2}(M) = 0$, so that $\Omega^2(M) = \Omega^{1,1}(M)$. Consider the statement $H^{0,1}_{\bar\partial}(\Delta) = 0$. Since $\Omega^{0,2}(\Delta) = 0$, any $(0,1)$-form is $\bar\partial$-closed, so we need to show that for any $g \in C^\infty(\Delta)$, the $(0,1)$-form $g(z,\bar z)d\bar z$ is in the image of $\bar\partial$, i.e. that there exists an $f \in C^\infty(\Delta)$ such that

$$g d\bar z = \bar\partial(f) = \frac{\partial f}{\partial \bar z} d\bar z.$$

This is equivalent to showing that there exists a solution to $\frac{\partial f}{\partial \bar z} = g$ for a given g. We first show this for compactly supported g.

Lemma 1.6.5. *Let g be a C^∞-function with compact support on $\mathbb{C}$. Then there exists a C^∞-function f on $\mathbb{C}$ such that $\frac{\partial f}{\partial \bar z} = g$. Moreover f is defined up to addition of a holomorphic function.*

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[5] If you have not seen the de Rham cohomology, it is defined the same way as Dolbeault cohomology, but using d instead of $\bar\partial$.

Proof. Set

$$f(z,\overline{z}) = -\frac{1}{2\pi i}\int\limits_{\mathbb{C}} \frac{g(\zeta,\overline{\zeta})}{\zeta - z}\, d\zeta\wedge d\overline{\zeta} \underset{\eta = z-\zeta}{=} \frac{1}{2\pi i}\int\limits_{\mathbb{C}} \frac{g(z-\eta,\overline{z}-\overline{\eta})}{\eta}\, d\eta\wedge d\overline{\eta}.$$

This converges for large η, since g has compact support. For small η, we rewrite in polar coordinates and get

$$\left|\frac{1}{2\pi i}\int\limits_{B(0,\varepsilon)} (\dots)\right| \leq C\int\limits_0^{\varepsilon}\int\limits_0^{2\pi} \frac{1}{r} r\; dr\; d\theta,$$

which converges. Hence f is well defined for all z, and we can write

$$f(z,\overline{z}) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int\limits_{\mathbb{C}\setminus\Delta_\varepsilon} \frac{g(z-\eta,\overline{z}-\overline{\eta})}{\eta}\, d\eta\wedge d\overline{\eta}.$$

The convergence is uniform, which means that we can differentiate under the integral and conclude that

$$\int\limits_{\mathbb{C}} \frac{1}{\eta}\frac{\partial^{i+j} g}{\partial x^i \partial y^j}(z-\eta,\overline{z}-\overline{\eta})\, d\eta\wedge d\overline{\eta}$$

converges, owing to the same argument as before. Therefore f is smooth. We now check that $\frac{\partial f}{\partial \overline{z}} = g$. First of all:

$$\frac{\partial f}{\partial \overline{z}} = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\mathbb{C}\setminus\Delta_\varepsilon} \frac{1}{\eta}\frac{\partial g}{\partial \overline{z}}(z-\eta,\overline{z}-\overline{\eta})\, d\eta\wedge d\overline{\eta}.$$

We can rewrite:

$$\frac{1}{\eta}\frac{\partial g}{\partial \overline{z}}(z-\eta,\overline{z}-\overline{\eta})\, d\eta\wedge d\overline{\eta} = -\frac{1}{\eta}\frac{\partial g}{\partial \overline{\eta}}(z-\eta,\overline{z}-\overline{\eta})\, d\eta\wedge d\overline{\eta} = d\left(\frac{g(z-\eta,\overline{z}-\overline{\eta})}{\eta}\, d\eta\right),$$

because $d(\varphi d\eta) = \frac{\partial\varphi}{\partial\overline{\eta}}\, d\overline{\eta}\wedge d\eta$. From the Stokes theorem we conclude:

$$\frac{1}{2\pi i}\int\limits_{\mathbb{C}\setminus\Delta_\varepsilon} \frac{1}{\eta}\frac{\partial g}{\partial \overline{z}}(z-\eta,\overline{z}-\overline{\eta})\, d\eta\wedge d\overline{\eta} = \frac{1}{2\pi i}\int\limits_{\mathbb{C}\setminus\Delta_\varepsilon} d\left(\frac{g(z-\eta,\overline{z}-\overline{\eta})}{\eta}\, d\eta\right)$$

$$= \frac{1}{2\pi i}\int\limits_{\partial\Delta_\varepsilon} \frac{g(z-\eta,\overline{z}-\overline{\eta})}{\eta}\, d\eta \underset{\eta=\varepsilon e^{i\theta}}{=} \frac{1}{2\pi i}\int\limits_0^{2\pi} \frac{g(z-\varepsilon e^{i\theta},\overline{z}-\varepsilon e^{-i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta$$

$$= \underbrace{\frac{1}{2\pi}\int\limits_0^{2\pi} g(z-\varepsilon e^{i\theta})d\theta}_{\text{average of g over the circle}} \xrightarrow{\varepsilon\to 0} g(z).$$

The second statement is obvious. ☐

Let now n be arbitrary.

Lemma 1.6.6. *Let $U \subset \mathbb{C}^n$ be an open polydisk and K a compact polydisk inside U. Let ω be a $\bar{\partial}$-closed (p,q)-form on U, $q \geq 1$. Then there exists an open polydisk V with $K \subset \overline{V} \subset U$ and a $(p,q-1)$-form θ on U, such that $\bar{\partial}\theta = \omega$ on V.*

Proof. We first reduce to the case $p = 0$. A (p,q)-form can be written as

$$\omega = \sum_{I=(i_1,\ldots,i_p)} \omega_I \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p},$$

where each ω_I is a $(0,q)$-form. Then:

$$\bar{\partial}\omega = \sum_I (\bar{\partial}\omega_I) \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

Hence ω is $\bar{\partial}$-closed if and only if each ω_I is $\bar{\partial}$-closed. Therefore, if the lemma is true for $p = 0$, then we can find polydisks V_I with $K \subset \overline{V}_I \subset U$ and $(0,q-1)$-forms θ_I s.t. $\bar{\partial}\theta_I = \omega_I$ on V_I. On $V = \bigcap_I V_I$ we then have $\bar{\partial}\theta = \omega$, where

$$\theta = \sum_I \theta_I \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

Thus we can assume that ω is a $\bar{\partial}$-closed $(0,q)$-form. We proceed by induction on the largest integer k such that $d\bar{z}_k$ appears in ω. If $k = 0$, then no $d\bar{z}_k$ appears, and $\omega = 0$ and we can take $\theta = 0$. Suppose that the claim holds for all integers $< k$ and let $\omega = \omega_0 + d\bar{z}_k \wedge \phi$, where both ω_0 and ϕ contain only $d\bar{z}_i$ with $i < k$. Write

$$\phi = \sum_{1 \leq \underbrace{j_1 \leq \cdots \leq j_{q-1}}_{=:J} < k} g_J \; d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_{q-1}}. \tag{1.6.1}$$

Observe that if $l > k$, then

$$\sum_J \frac{\partial g_J}{\partial \bar{z}_l} \; d\bar{z}_l \wedge d\bar{z}_k \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_{q-1}}$$

is the only term containing $d\bar{z}_l \wedge d\bar{z}_k \wedge \bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_{q-1}}$ in $\bar{\partial}\omega$. Therefore $\frac{\partial g_J}{\partial \bar{z}_l} = 0$ for $l > k$, so that each g_J is holomorphic in $z_{k+1}, \ldots, z_n$. We can multiply ω by a bump function, compactly supported inside U and equal to 1 on an open polydisk V such that $K \subset \overline{V} \subset U$. According to Lemma 1.6.5, we can find functions f_J such that $\frac{\partial f_J}{\partial \bar{z}_k} = g_J$ for each J occuring in (1.6.1). Since the f_J are defined by integrating with respect to $\bar{z}_k$, they remain holomorphic in $z_{k+1}, \ldots, z_n$. Set

$$\alpha = \sum_J f_J \; d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_{q-1}},$$

where the summation is over the same J as in (1.6.1). Then $d\bar{z}_k \wedge \phi - \bar{\partial}\alpha$ contains only $d\bar{z}_i$ with $i < k$ (on V). The same is then true for $\omega - \bar{\partial}\alpha$. From the inductive assumption, $\omega - \bar{\partial}\alpha = \bar{\partial}\beta$ on some smaller polydisk, which means that $\omega = \bar{\partial}(\alpha + \beta)$. □

Proof of the $\bar{\partial}$-Poincaré lemma for arbitrary n. Let U_i, $i \in \mathbb{N}$, be a monotone increasing sequence of polydisks such that $\bigcup U_i = \Delta$ and each $\overline{U}_i$ is compact, with $\overline{U}_i \subset U_{i+1}$. As in the proof of lemma 1.6.6, we only need to consider $p = 0$. Owing to that lemma, there exist $\theta_i \in \Omega^{0,q-1}(\Delta)$ such that $\bar{\partial}\theta_i = \omega$ on U_i. We need to show that we can choose θ_i in such a way that they converge to a $(0, q)$-form θ on Δ. We proceed by induction on q.

If $q = 1$, then θ_i are smooth functions on Δ with $\bar{\partial}\theta_i = \omega$ on U_i. If $\alpha \in C^\infty(\Delta)$ satisfies $\bar{\partial}\alpha = \omega$ in U_{i+1} (e.g. $\alpha = \theta_{i+1}$), then $\bar{\partial}(\theta_i - \alpha) = 0$ in U_i, so $\theta_i - \alpha$ is holomorphic on U_i and hence has a power series expansion around 0. We can truncate to obtain a (holomorphic) polynomial β with

$$\sup_{U_{i-1}} |(\theta_i - \alpha) - \beta| < \frac{1}{2^i}.$$

Since β is a polynomial, it is holomorphic on $\mathbb{C}^n$. Set $\theta_{i+1} = \alpha + \beta$ on U_{i+1}. Then

$$\bar{\partial}\theta_{i+1} = \bar{\partial}\alpha = \omega \quad \text{in} \quad U_{i+1} \quad \text{and} \quad \sup_{U_{i-1}} |\theta_{i-1} - \theta_i| < \frac{1}{2^i}.$$

Therefore $(\theta_j)_{j \geq 1}$ is a Cauchy sequence on each $\overline{U}_{i-1}$, so that (θ_j) converges on compact subsets. We obtain a θ with $\bar{\partial}\theta = \omega$.

For $q \geq 2$ we proceed similarly. Take $\alpha \in \Omega^{0,q-1}(\Delta)$ with $\bar{\partial}\alpha = \omega$ on U_{i+1} (e.g. $\alpha = \theta_{i+1}$), so that $\bar{\partial}(\theta_i - \alpha) = 0$ on U_i. Since $\theta_i - \alpha \in Z^{0,q-1}(\Delta)$, the inductive hypothesis implies that there exists a $\psi \in \Omega^{0,q-2}(\Delta)$ with $\bar{\partial}\psi = \theta_i - \alpha$ in U_{i-1}. Set $\theta_{i+1} = \alpha + \bar{\partial}\psi$. Then $\bar{\partial}\theta_{i+1} = \bar{\partial}\alpha = \omega$ in U_i and $\theta_{i+1} = \theta_i$ on U_{i-1}. It follows that the θ_i converge uniformly on compact sets. □

Using annuli and Laurent series expansions one can show similarly that

$$H^{p,q}_{\bar{\partial}}\big((\Delta^*)^k \times \Delta^l\big) = 0 \quad \forall q \geq 1, p \geq 0,$$

where Δ^* is the punctured disk in $\mathbb{C}$. This is, however, false for $\Delta^2 \backslash \{\mathrm{pt}\}$:

Example 1.6.7. We shall show that $\dim H^{0,1}_{\bar{\partial}}(\mathbb{C}^2 \backslash \{0\}) = \infty$. Observe that $\mathbb{C}^2 \backslash \{0\}$ is homotopy equivalent to S^3, so this example shows that the Dolbeault cohomology is no a topological invariant, unlike the de Rham cohomology[6]. Let

$$U_1 = \{z_1 \neq 0\} = \mathbb{C}^* \times \mathbb{C} \quad \text{and} \quad U_2 = \{z_2 \neq 0\} = \mathbb{C} \times \mathbb{C}^*,$$

so that $\mathbb{C}^2 \backslash \{0\} = U_1 \cup U_2$ and $U_1 \cap U_2 = \mathbb{C}^* \times \mathbb{C}^*$. Let λ_1, λ_2 be a partition of unity subordinate to $\{U_1, U_2\}$ and let f be a holomorphic function on $U_1 \cap U_2$. Then $g_1 = \lambda_2 f$ is a smooth function on U_1 and $g_2 = -\lambda_1 f$ is a smooth function

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[6] Actually, already Remark 1.6.4 shows this.

on U_2. On $U_1 \cap U_2$ we have $f = g_1 - g_2$, so that $\bar{\partial}(g_1 - g_2) = \bar{\partial} f = 0$ and we can define a $(0,1)$-form on $\mathbb{C}^2 \backslash \{0\}$ by

$$\omega = \begin{cases} \bar{\partial} g_1 = f \bar{\partial} \lambda_2 & \text{on } U_1 \\ \bar{\partial} g_2 = -f \bar{\partial} \lambda_1 & \text{on } U_2. \end{cases}$$

Clearly $\bar{\partial}\omega = 0$. Suppose that $\omega = \bar{\partial} h$ for some $h \in C^\infty(\mathbb{C}^2 \backslash \{0\})$. Then $\bar{\partial}(g_1 - h) = 0$ on U_1 and $\bar{\partial}(g_2 - h) = 0$ on U_2. Hence $(g_1 - h)$ is holomorphic on U_1 and $(g_2 - h)$ is holomorphic on U_2. But then $f = (\lambda_1 + \lambda_2) f = g_1 - g_2 = (g_1 - h) - (g_2 - h)$, which means that $f = u_1 + u_2$, where u_1 is holomorphic on U_1 and u_2 is holomorphic on U_2. Consider the Laurent series of u_1 and u_2:

$$u_1 = \sum_{\substack{j \geq 0 \\ i \in \mathbb{Z}}} \alpha^1_{ij} z_1^i z_2^j \qquad u_2 = \sum_{\substack{i \geq 0 \\ j \in \mathbb{Z}}} \alpha^2_{ij} z_1^i z_2^j,$$

and observe that the sum $u_1 + u_2$ does not have any terms of the form $z_1^{-m} z_2^{-n}$ with $m, n > 0$. Therefore the $\bar{\partial}$-closed form ω defined by $f = z_1^{-m} z_2^{-n}$ is not $\bar{\partial}$-exact[7].

This example shows that usually there is no relation between the Dolbeault and the de Rham cohomology groups. Nor should there be: solving the equation $d\alpha = \beta$ is very different from solving $\bar{\partial}\alpha = \beta$. We shall see later a true miracle: for *projective manifolds* these two cohomology theories are very closely related.

Further reading:
There is an important class of complex manifolds with trivial Dolbeault cohomology: the so-called *Stein manifolds*. These are biholomorphic to complex submanifolds of $\mathbb{C}^N$, and are, in a sense, an exact opposite of compact complex manifolds: they have plenty of global holomorphic functions. You can read up on the definition and basic properties of Stein manifolds (but not yet on their Dolbeault cohomology) in §I.6 of Demailly's book "Complex Analytic and Differential Geometry", freely available online at:
`https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`

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[7] A form is called $\bar{\partial}$-exact if it belongs to $\operatorname{Im} \bar{\partial}$.

Chapter 2

Vector bundles and sheaves

2.1 Complex and holomorphic vector bundles

Let M be a smooth manifold. A (smooth) *complex vector bundle of rank k* on M consists of a family $\{E_x\}_{x\in M}$ of k-dimensional complex vector spaces parametrised by M, together with a C^∞-manifold structure on

$$E = \bigsqcup_{x\in M} E_x$$

such that

1) the projection $\pi : E \to M$, $\pi(E_x) = \{x\}$, is C^∞ and

2) each point $x_0 \in M$ has an open neighborhood U, such that there exists a diffeomorphism
$$\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k,$$
which maps the vector space E_x isomorphically[1] onto $\{x\} \times \mathbb{C}^k$ for each $x \in U$.

The map φ_U is called a *trivialisation* of E over U. The vector spaces E_x are called the *fibres* of E. A vector bundle of rank 1 is called a *line bundle.*

Examples 2.1.1. (i) The complexified tangent bundle of a smooth manifold;
(ii) if M is almost complex, then $T^{1,0}M, T^{0,1}M, \Lambda^{p,q}M$, etc.

For any pair φ_U, φ_V of local trivialisations, we obtain a C^∞-map

$$g_{UV} : U \cap V \to GL(k, \mathbb{C})$$

given by
$$g_{UV}(x) = \big(\varphi_U \circ \varphi_V^{-1}\big)|_{\{x\}\times\mathbb{C}^k}.$$

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[1] meaning an isomorphism of vector spaces

These *transition functions* satisfy

$$\begin{aligned} &g_{UV}(x)g_{VU}(x) = 1 \\ &g_{UV}(x)g_{VW}(x)g_{WU}(x) = 1. \end{aligned}$$

Conversely, given an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M and C^∞-maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{C})$$

satisfying these identities, there exits a unique (up to an isomorphism) complex vector bundle $E \overset{\pi}{\to} M$ with transition functions $\{g_{\alpha\beta}\}$:

$$E = \bigsqcup_{x \in M} U_\alpha \times \mathbb{C}^k / \sim,$$

where

$$(\alpha, x, v) \sim (\beta, y, w) \Longleftrightarrow x = y \quad \text{and} \quad v = g_{\alpha\beta}(x)w.$$

Any operation on vector spaces induces an operation on vector bundles by performing it a each point $x \in M$. Thus, given two vector bundles E and F on M, we can construct:

- the dual bundle E^*
- direct sum of vector bundles $E \oplus F$
- tensor product $E \otimes F$
- exterior powers $\bigwedge^r E$

The corresponding transition functions are easy to determine: if E and F have ranks k and l, and transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, respectively, then the transition functions of $E^*, E \oplus F, E \otimes F$ are:

$$\left((g_{\alpha\beta})^T\right)^{-1}, \quad \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \in GL(\mathbb{C}^k \oplus \mathbb{C}^l), \quad g_{\alpha\beta} \otimes h_{\alpha\beta} \in GL(\mathbb{C}^k \otimes \mathbb{C}^l).$$

An important example is the *determinant bundle* $\det E = \bigwedge^k E$ of E ($k = \operatorname{rank} E$). It is a line bundle with transition functions $\det(g_{\alpha\beta})(x) \in GL(1, \mathbb{C}) \simeq \mathbb{C}^*$.

A *subbundle* $F \subset E$ of a vector bundle is a smooth submanifold F of E such that $\pi^{-1}(x) \cap F$ is a (complex) vector subspace for each $x \in M$. This means that there exists a family of local trivialisations of E, relative to which the transition functions look as follows

$$g_{UV}(x) = \begin{pmatrix} h_{UV}(x) & k_{UV}(x) \\ 0 & j_{UV}(x) \end{pmatrix},$$

where h_{UV} are the transition functions for F. Observe that j_{UV} are the transition functions of the quotient bundle E/F.

A *homomorphism* between vector bundles E on M and F on N is given by a C^∞-map $f : E \to F$ such that $f|_{E_x}$ maps E_x linearly to $F_{f(x)}$. Observe that we could define $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$, but these will not in general be subbundles of E or F, since the rank of $f|_{E_x}$ may vary. In fact, it is better not to do this, since a monomorphism between vector bundles can have a nonzero $\mathrm{Ker}(f)$ in this sense.

A vector bundle E on M is called *trivial* if E is isomorphic to the product bundle $M \times \mathbb{C}^k$.

Given a C^∞-map $f : M \to N$ and a vector bundle $F \xrightarrow{\pi} N$, we define the *pullback bundle* f^*F on M by

$$f^*F = \{(x, w) \in M \times F \mid f(x) = \pi(w)\}, \quad \text{i.e.} \quad (f^*F)_x = F_{f(x)}.$$

A *section* s of a vector bundle $E \xrightarrow{\pi} M$ is a C^∞-map $s : M \to E$ such that $s(x) \in E_x$ for all $x \in M$ (just like a vector field). The vector space of sections is denoted by $\Gamma(E)$.

Observe that trivialising a rank k bundle E over an open subset $U \subset M$ is equivalent to giving k sections $s_1, \dots, s_k$, which are linearly independent at every point of U. Such a collection $s_1, \dots, s_k$ is called a *frame* for E over U.

Let now M be a complex manifold.

A *holomorphic vector bundle* $E \xrightarrow{\pi} M$ is a complex vector bundle with holomorphic transition functions. This implies in particular that E is a complex manifold and $\pi : E \to M$ is holomorphic.

Examples 2.1.2. 1) The holomorphic tangent bundle TM ($\simeq T^{1,0}M$);

2) $\Lambda^{p,0}M$ for $p \geq 1$ (but not $\Lambda^{p,q}M$ for $q \neq 0$). The sections of $\Lambda^{p,0}M$ are holomorphic p-forms.

3) The line bundle $\Lambda^{n,0}M$, where $n = \dim_{\mathbb{C}} M$, is called the *canonical bundle* of M and is denoted by K_M. It's dual $K_M^* = \Lambda^n(T^{1,0}M)$ is called the *anti-canonical bundle.*

4) The tautological line bundle over $\mathbb{CP}^m$ is a complex line bundle $\pi : J \to \mathbb{CP}^m$, with the fibre $J_{[z]}$ over $[z] \in \mathbb{CP}^m$ being the line $\langle z \rangle$ in $\mathbb{C}^{m+1}$. Recall the standard atlas $(U_i, \varphi_i)_{i=0,\dots,m}$ of $\mathbb{CP}^m$. The corresponding trivialisations of J is:

$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}; \quad \psi_i([z], w) = ([z], w_i).$$

The transition functions are

$$\psi_i \circ \psi_j^{-1}([z], \lambda) = \psi_i\left([z], \lambda\frac{z}{z_j}\right) = \left([z], \lambda\frac{z_i}{z_j}\right),$$

so that $g_{ij}([z]) = \frac{z_i}{z_j}$. Therefore J is a holomorphic bundle.

Remark 2.1.3. If E is a holomorphic vector bundle over a complex manifold, we have to distinguish between its smooth sections and its holomorphic sections. The space of smooth sections is denoted by $\Gamma(E)$; the space of holomorphic sections by $H^0(M, E)$ - this notation will be explained somewhat later.

Proposition 2.1.4.

$$K_{\mathbb{CP}^m} \simeq J^{m+1},$$

i.e. the canonical bundle of $\mathbb{CP}^m$ *is isomorphic to the* $(m+1)$*-th (tensor) power of the tautological bundle.*

Proof. We consider the dual bundle $H = J^*$, called the *hyperplane bundle*. The fibre $H_{[z]}$ consists of linear maps $\langle z \rangle \to \mathbb{C}$. Recall that

$$\mathbb{CP}^m \simeq \mathbb{C}^{m+1} \backslash \{0\} / \mathbb{C}^*$$

On $\mathbb{C}^{m+1} \backslash \{0\}$ we have the (holomorphic) vector fields $\frac{\partial}{\partial x_i}$, $i = 0, \ldots, m$, where x_i are coordinates on $\mathbb{C}^{m+1}$. If $v_0, \ldots, v_m$ are linear functionals on $\mathbb{C}^{m+1}$, then the vector field

$$\sum_{i=0}^{m} v_i \frac{\partial}{\partial x_i}$$

is $\mathbb{C}^*$-invariant, and so it defines a holomorphic vector field on $\mathbb{CP}^m$. It is easy to see that these vector fields generate $T_{[z]}\mathbb{CP}^m$ at each $[z]$. Hence we get a surjective map between vector bundles

$$H \otimes \mathbb{C}^{m+1} \longrightarrow T\mathbb{CP}^m \to 0.$$

The kernel corresponds to the radial vector field $E = \sum x_i \frac{\partial}{\partial x_i}$ on $\mathbb{CP}^{m+1}$ (this is the vector field on $\mathbb{C}^{m+1}$ tangent to orbits of $\mathbb{C}^*$, hence inducing 0 in $T\mathbb{CP}^m$). Thus we have an exact[2] sequence

$$0 \to \underline{\mathbb{C}} \longrightarrow H \otimes \mathbb{C}^{m+1} \longrightarrow T\mathbb{CP}^m \to 0,$$

where the $\underline{\mathbb{C}} \simeq \mathbb{CP}^m \times \mathbb{C}$ denotes the trivial line bundle on $\mathbb{CP}^m$ (generated by the vector field E). Now observe that for an exact sequence of vector spaces $0 \to U \to V \to W \to 0$, of dimensions k, n, l respectively, $\Lambda^n V \simeq \Lambda^k U \otimes \Lambda^l W$ (since $V \simeq U \oplus W$). Applying this isomorphism pointwise, we obtain $K^*_{\mathbb{CP}^m} \simeq H^{m+1}$. □

Example 2.1.5 ($\mathbb{CP}^1$). The standard atlas of $\mathbb{CP}^1 \simeq \{[x_0, x_1]; x_0, x_1 \in \mathbb{C}\}$ consists of $U_0 = \{[x_0, x_1]; x_0 \neq 0\}$ and $U_1 = \{[x_0, x_1]; x_0 \neq 1\}$. The corresponding coordinates are $\zeta = x_1/x_0$ on U_0 and $\tilde{\zeta} = x_0/x_1$ on U_1, so that $\tilde{\zeta} = 1/\zeta$. The transition function of the tautological bundle J from U_0 to U_1 is ζ. The (holomorphic) cotangent bundle $T^*\mathbb{P}^1$ is trivialised by sections $d\zeta$ on U_0 and $d\tilde{\zeta}$ on U_1. Since $d\tilde{\zeta} = d(\zeta^{-1}) = -\zeta^{-2} d\zeta$, the transition function for $T^*\mathbb{P}^1 = K_{\mathbb{CP}^1}$

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[2] A sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ is called exact, if f is injective, g is surjective, and $\operatorname{Ker} g = \operatorname{Im} f$.

from U_0 to U_1 is $-\zeta^2$. Changing the sign of the transition function gives an isomorphic vector bundle, and hence $K_{\mathbb{CP}^1} \simeq J^2$.

$\mathbb{CP}^1$ is one of the very few projective manifolds on which vector bundles can be classified[3]. The Birkhoff-Grothendieck theorem[4] says that any (holomorphic) vector bundle E on $\mathbb{P}^1$ splits into a direct sum of line bundles, i.e.

$$E \simeq H^{k_1} \oplus \cdots \oplus H^{k_r},$$

where H is the hyperplane bundle and $r = \operatorname{rank} E$. Let us see how one may prove this. The bundle E can be trivialised over U_0 and U_1 (this is perhaps not quite obvious yet, but let us assume it). The transition function is then a holomorphic map $U_0 \cap U_1 \to GL(r, \mathbb{C})$. In terms of the affine coordinate ζ introduced above it is a holomorphic map $\mathbb{C}^* \to GL(r, \mathbb{C})$. We expand this map in Laurent series, so that the transition function is an $r \times r$ matrix $g(\zeta, \zeta^{-1})$ with entries given by Laurent series and nonvanishing determinant for $\zeta \neq 0, \infty$. This determinant is the transition function for $\det E$, which is a line bundle, and hence isomorphic to H^n, for some n (this is also something to prove later). On the other hand, the transition function of $H^{k_1} \oplus \cdots \oplus H^{k_r}$ is the diagonal matrix $\operatorname{diag}(\zeta^{-k_1}, \ldots, \zeta^{-k_r})$. Two vector bundles are isomorphic, if there exist holomorphic changes of trivialisations $g_0 : U_0 \to GL(r, \mathbb{C})$, $g_1 : U_1 \to GL(r, \mathbb{C})$. Therefore the statement of the Birkhoff-Grothendieck theorem is equivalent to the following special case of *Birkhoff's factorisation*:

An invertible matrix $g(\zeta, \zeta^{-1})$ with entries that are Laurent polynomials and determinant equal to ζ^n for some $n \in \mathbb{Z}$ can be factorised as

$$g_1(\zeta^{-1}) \operatorname{diag}\big(\zeta^{-k_1}, \ldots, \zeta^{-k_r}\big)\, g_0(\zeta),$$

where g_0 (resp. g_1) is holomorphic in ζ (resp. in ζ^{-1}) with constant determinant, and $k_1, \ldots, k_r \in \mathbb{Z}$.

Thus the Birkhoff-Grothendieck theorem reduces to this purely algebraic statement. See "further reading" below for references containing a proof of this.

Example 2.1.6 (*Tautological bundle on a Grassmannian).* Recall from §1.2 that the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ parametrises k-dimensional subspaces of $\mathbb{C}^n$. Just as for $\mathbb{CP}^m$ $(k = 1)$ we can define a complex vector bundle $\mathcal{U}_{k,n}$ over $\mathrm{Gr}_k(\mathbb{C}^n)$ by attaching to each point the k-dimensional subspace which defines it. Again, this is a holomorphic vector bundle. It is also a subbundle of the trivial bundle $\underline{\mathbb{C}}^n$ and we obtain a short exact sequence of vector bundles on $\mathrm{Gr}_k(\mathbb{C}^n)$:

$$0 \to \mathcal{U}_{k,n} \longrightarrow \underline{\mathbb{C}}^n \longrightarrow \mathcal{Q}_{k,n} \to 0.$$

Observe that the fibre of the quotient bundle $\mathcal{Q}_{k,n}$ at a $[W] \in \mathrm{Gr}_k(\mathbb{C}^n)$ is the quotient vector space $\mathbb{C}^n/W$. In particular, $\mathcal{Q}_{k,n}$ has rank $n - k$.
Recall now the description of $\mathrm{Gr}_k(\mathbb{C}^n)$ as the homogenous manifold $GL(n, \mathbb{C})/H$,

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[3] The others are *elliptic curves*, i.e. tori of the form $\mathbb{C}/\Lambda$, where Λ is a lattice of full rank.

[4] Equivalent statements were actually proved much earlier by Kronecker and by Dedekind and Weber.

where H is the subgroup stabilising $S_0 = \langle e_1, \dots, e_k \rangle$. H acts on S_0 and $\mathcal{U}_{k,n} \simeq \big(GL(n,\mathbb{C}) \times S_0\big)/H$ - just observe that the injection $\mathcal{U}_{k,n} \to \underline{\mathbb{C}}^n$ is induced by $GL(n,\mathbb{C}) \times S_0 \ni (g,v) \mapsto gv$. Similarly $\mathcal{Q}_{k,n} \simeq \big(GL(n,\mathbb{C}) \times (\mathbb{C}^n/S_0)\big)/H$. Now consider the (holomorphic) tangent bundle of the Grassmannian. Recall that $TGL(n,\mathbb{C}) \simeq GL(n,\mathbb{C}) \times \mathfrak{gl}(n,\mathbb{C})$ and that the right translations by $GL(n,\mathbb{C})$ on $TGL(n,\mathbb{C})$ corresponds to the adjoint action on $\mathfrak{gl}(n,\mathbb{C})$. Denote by $\mathfrak{m}$ the subspace complementary to $\mathrm{Lie}(H)$ in $\mathfrak{gl}(n,\mathbb{C})$, i.e.

$$\mathfrak{m} = \underbrace{\left(\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right)}_{k \quad n-k} \begin{array}{l} \}k \\ \}n-k \end{array}$$

Then $T\mathrm{Gr}_k(\mathbb{C}^n) \simeq \big(GL(n,\mathbb{C}) \times \mathfrak{m}\big)/H$. On the other hand $\mathfrak{m} \simeq S_0^* \otimes (\mathbb{C}^n/S_0)$ (linear homomorphisms from S_0 to $\mathbb{C}^n/S_0$). Taken together, this shows that

$$T\mathrm{Gr}_k(\mathbb{C}^n) \simeq \mathcal{U}_{k,n}^* \otimes \mathcal{Q}_{k,n}.$$

Remark 2.1.7. $\mathcal{U}_{k,n}$ is also called the *universal bundle*, since every complex vector bundle of rank k over a compact manifold M is the pullback of $\mathcal{U}_{k,n}$ with respect to a smooth map $f : M \to \mathrm{Gr}_k(\mathbb{C}^n)$ for some n large enough.

Further reading:

i) For a proof of the Birkhoff-Grothendieck theorem see:
M. Hazewinkel and C.F. Martin, *A short elementary proof of Grothendieck's theorem on algebraic vector bundles over the projective line*, Journal of Pure and Applied Algebra 25 (1982), 207–211.
Note that this paper proves the factorisation mentioned in Example 2.1.5 only for Laurent *polynomials*, so for *algebraic* vector bundles. This *is* equivalent to the classification of *holomorphic* vector bundles, due to a fundamental result, called GAGA[5], which states that on projective manifolds *"holomorphic"="algebraic"*.

ii) Birkhoff's factorisation and its generalisations are a huge area by themselves with close links to loop groups, Kac-Moody algebras, integrable systems, operator theory, and more. For a very down to earth approach, take a look at the book "Factorization of matrix functions and singular integral operators" by K. Clancey and I. Gohberg (Springer 1981).

iii) For vector bundles on higher dimensional projective spaces, the book "Vector bundles on complex projective spaces" by Okonek, Schneider, and Spindler (Birkhäuser 1980) is still a very valuable reference. It will be, however, easier to read once we cover sheaves.

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[5] J.-P. Serre, *Géometrie algébrique et géometrie analytique*, Annales Inst. Fourier 6 (1956), 1–42

iv) Grassmannians are a special case of the so-called *flag manifolds*. A brief introduction (with necessary references) may be found in §3.1 of "Lie group actions in complex analysis" by D.N. Akhiezer (Vieweg 1995). It does require a background in Lie theory, though.

Vector bundles on flag manifolds have many applications. One of the most important is a geometric construction of finite-dimensional representations of complex semisimple Lie groups; see chapter 4 of the same book (again, if you are not familiar with sheaves, better wait a week or so).

2.2 Pseudoholomorphic structures on complex vector bundles

Let E be a complex vector bundle over an almost complex manifold M. For every p, q we consider the vector bundle

$$\Lambda^{p,q}(E) = \Lambda^{p,q}M \otimes E$$

and denote the space of its sections by $\Omega^{p,q}(E)$ - these are called *E-valued differential forms of type* (p, q). If we choose a local trivialisation of E, i.e. a local frame $(e_1 \ldots, e_k)$, then $\sigma \in \Omega^{p,q}(E)$ can be written in this trivialisation as

$$\sigma = (\omega_1, \ldots, \omega_k) = \sum_{i=1}^{k} \omega_i \otimes e_i,$$

where ω_i are local (p, q)-forms on M.

Suppose now that M is complex and E is holomorphic. Let (e_i) be a holomorphic frame. It turns out that the operator

$$\begin{aligned} &\bar{\partial} : \ \Omega^{p,q}(E) \to \Omega^{p,q+1}(E) \\ &(\omega_1, \ldots, \omega_k) \longmapsto (\bar{\partial}\omega_1, \ldots, \bar{\partial}\omega_k) \end{aligned}$$

is well defined, i.e. it does not depend on the trivialisation. Indeed, if $(e'_1, \ldots, e'_k)$ is another holomorphic frame with $e_i = \sum\limits_{j=1}^{k} g_{ij} e'_j$, where the g_{ij} are holomorphic, then

$$\sigma = \sum_{i=1}^{k} \omega_i \otimes e_i = \sum_{j=1}^{k} \left(\sum_{i=1}^{k} g_{ij} \omega_i \right) e'_j.$$

Hence in the new frame

$$\bar{\partial}\sigma = \sum_{j=1}^{k} \bar{\partial} \left(\sum_{i=1}^{k} g_{ij} \omega_i \right) e'_j = \sum_{i,j=1}^{k} (g_{ij} \bar{\partial}\omega_i) \otimes e'_j = \sum_{i=1}^{k} (\bar{\partial}\omega_i) \otimes e_i.$$

Observe now that $\bar{\partial}$ satisfies $\bar{\partial}^2 = 0$ and the Leibniz rule

$$\bar{\partial}(\omega \wedge \sigma) = \bar{\partial}\omega \wedge \sigma + (-1)^{r+s}\omega \wedge \bar{\partial}\sigma, \quad \omega \in \Omega^{r,s}(M),\ \sigma \in \Omega^{p,q}(E).$$

The existence of such a natural operator $\bar{\partial}$ on E-valued forms is a remarkable property of holomorphic vector bundles. In fact, it characterises holomorphic vector bundles among complex vector bundles, as we shall shortly see.

Definition 2.2.1. Let E be a complex vector bundle on a complex manifold M. An operator

$$\bar{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$$

satisfying the Leibniz rule is called a *pseudo-holomorphic structure* on E. If, in addition, $\bar{\partial}^2 = 0$, then $\bar{\partial}$ is called a *holomorphic structure*[6]. A section s of a pseudo-holomorphic vector bundle $(E, \bar{\partial})$ is called $\bar{\partial}$-holomorphic if $\bar{\partial}s = 0$.

Remark 2.2.2. The Leibniz rule implies that $\bar{\partial}$ is determined by its action $\bar{\partial} : \Gamma(E) \to \Omega^{0,1}(E)$ on $\Gamma(E) = \Omega^{0,0}(E)$.

Theorem 2.2.3. *A complex vector bundle E is holomorphic if and only if it admits a holomorphic structure $\bar{\partial}$.*

Proof. The idea is to use $\bar{\partial}$ to define an almost complex structure J on E, linear on fibres, so that the projection $E \xrightarrow{\pi} M$ is an almost complex map. Then we shall show that $\bar{\partial}^2 = 0$ if and only if J is integrable.

Lemma 2.2.4. *A pseudo-holomorphic vector bundle $(E, \bar{\partial})$ of rank k is holomorphic if and only if every point of M has a neighbourhood with a $\bar{\partial}$-holomorphic frame.*

Remark 2.2.5. Compare with lemma 1.5.13.

Proof. If E is holomorphic, then $\bar{\partial}$-holomorphic is the same as holomorphic in the usual sense, so $\bar{\partial}$-holomorphic frames exist. Conversely, suppose that $\bar{\partial}$-holomorphic frames exist and let e, e' be two such frames on U, U'. On $U \cap U'$ we can write $e_i' = \sum g_{ij} e_j$ and then, using the Leibniz rule,

$$0 = \bar{\partial}e_i' = \sum_{j=1}^{k} \bar{\partial}g_{ij} e_j + \sum_{j=1}^{k} g_{ij} \bar{\partial}e_j = \sum_{j=1}^{k} \bar{\partial}g_{ij} e_j.$$

Hence $\bar{\partial}g_{ij} = 0$, and therefore the g_{ij} are holomorphic transition functions. □

Proof of Theorem 2.2.3. The "only if" part has been already shown. Suppose that E has a holomorphic structure $\bar{\partial}$. We need to show that there exists a $\bar{\partial}$-holomorphic frame around each $x \in M$. Let $(\sigma_1, \ldots, \sigma_k)$ be an arbitrary smooth local frame around x and define local $(0,1)$-forms τ_{ij} by

$$\bar{\partial}\sigma_i = \sum_{j=1}^{k} \tau_{ij} \otimes \sigma_j.$$

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[6] Or a *Dolbeault operator*.

The assumption $\bar{\partial}^2 = 0$ yields

$$0 = \bar{\partial}^2 \sigma_i = \sum_{j=1}^{k} \bar{\partial}\tau_{ij} \otimes \sigma_j - \sum_{j,l=1}^{k} \tau_{il} \wedge \tau_{lj} \otimes \sigma_j .$$

Hence

$$\bar{\partial}\tau_{ij} = \sum_{l=1}^{k} \tau_{il} \wedge \tau_{lj} \quad \forall i, j = 1, \dots, k.$$

We seek a $\bar{\partial}$-holomorphic frame $(e_1, \dots, e_k)$. It can be written as $e_i = \sum\limits_{j=1}^{k} f_{ij}\sigma_j$ for some local functions f_{ij}. Then

$$0 = \bar{\partial} e_i = \sum_{j=1}^{k} \bar{\partial} f_{ij} \otimes \sigma_j + \sum_{j=1}^{k} f_{ij} \bar{\partial}\sigma_j = \sum_{j=1}^{k} \left(\bar{\partial} f_{ij} + \sum_{l=1}^{k} f_{il}\tau_{lj} \right) \otimes \sigma_j, \quad i = 1, \dots, k.$$

We can write this as an equation on matrices $f = (f_{ij})$, $\tau = (\tau_{ij})$:

$$\bar{\partial} f + f \cdot \tau = 0. \tag{2.2.1}$$

This is the equation we need to solve for f.

We may suppose that we are on an open subset U of $\mathbb{C}^m$ with holomorphic coordinates z_α and $E|_U \simeq U \times \mathbb{C}^k = Y$ with coordinates $w_1, \dots, w_k$ on $\mathbb{C}^k$. Consider the subbundle T of $\Lambda^1 Y \otimes \mathbb{C}$ (the complexified cotangent bundle) generated by 1-forms

$$\left\{ dz_\alpha, dw_i - \sum_{l=1}^{k} \tau_{il} w_l \right\}_{\substack{l \leq \alpha \leq m \\ 1 \leq i \leq k}} .$$

Let T' be defined the same way, but with everything conjugated. Then $\Lambda^1 Y \otimes \mathbb{C} \simeq T \oplus T'$, and setting $J = i$ on T, $J = -i$ on T', defines an almost complex structure on Y such that $\Lambda^{1,0} Y = T$. We claim that this almost complex structure is integrable. Owing to Proposition 1.5.4 this is equivalent to

$$d\Omega^{1,0} \subset \Omega^{2,0} \oplus \Omega^{1,1}, \quad \text{i.e.} \quad d\left(\Gamma(T)\right) \subset \Gamma\left(T \wedge \Omega^1_{\mathbb{C}}(Y)\right).$$

Clearly $d(dz_\alpha) = 0$ and

$$\begin{aligned} d\left(dw_i - \sum_{l=1}^{k} \tau_{il} w_l \right) &= -\sum_{l=1}^{k} (\partial \tau_{il}) w_l - \sum_{l=1}^{k} (\bar{\partial}\tau_{il}) w_l + \sum_{l=1}^{k} \tau_{il} \wedge dw_l \\ &= -\sum_{l=1}^{k} (\partial \tau_{il}) w_l - \sum_{s,l=1}^{k} (\tau_{is} \wedge \tau_{sl}) w_l + \sum_{l=1}^{k} \tau_{il} \wedge dw_l \\ &= -\sum_{l=1}^{k} (\partial \tau_{il}) w_l + \sum_{s=1}^{k} \tau_{is} \wedge \left(dw_s - \sum_{l=1}^{k} \tau_{sl} w_l \right). \end{aligned}$$

The terms in the second sum clearly belong to $\Gamma\left(T \wedge \Omega^1_{\mathbb{C}}(Y)\right)$. The terms in the first sum, $(\partial \tau_{il})w_l$, are forms of type $(1,1)$ on U, and hence also belong to $\Gamma\left(T \wedge \Omega^1_{\mathbb{C}}(Y)\right)$, since $dz_\alpha \in \Gamma(T)$. Therefore the almost complex structure is integrable, and we have local complex coordinates z_α, u_i, $\alpha = 1, \dots, n$, $i = 1, \dots, k$, in some neighbourhood of $(x, 0)$ in $U \times \mathbb{C}^k$. In particular $du_i \in \Gamma(T)$, i.e.

$$du_i = \sum_{j=1}^{k} F_{ij} \left(dw_j - \sum_{s=1}^{k} \tau_{js} w_s \right) + \sum_{\alpha=1}^{n} G_{i\alpha} dz_\alpha$$

for some smooth functions $F_{ij}, G_{i\alpha}$. Taking the exterior derivative of both sides gives

$$0 = \sum_{j=1}^{k} dF_{ij} \wedge \left(dw_j - \sum_{s=1}^{k} \tau_{js} w_s \right) + \sum_{j,s=1}^{k} F_{ij} \left(-d\tau_{js} w_s + \tau_{js} \wedge dw_s \right) + \sum_{\alpha=1}^{n} dG_{i\alpha} \wedge dz_\alpha.$$

If we now set $w_1, \dots, w_k = 0$, then

$$\begin{aligned} 0 &= \sum_{j=1}^{k} dF_{ij}(z,0) \wedge dw_j + \sum_{j=1}^{k} F_{ij}(z,0) \sum_{s=1}^{k} \tau_{js} \wedge dw_s + \sum_{\alpha=1}^{n} dG_{i,\alpha}(z,0) \wedge dz_\alpha = \\ &= \sum_{j=1}^{k} \left(dF_{ij}(z,0) + \sum_{l=1}^{k} F_{il}(z,0) \tau_{lj} \right) \wedge dw_j + \sum_{\alpha=1}^{n} dG_{i\alpha}(z,0) \wedge dz_\alpha. \end{aligned}$$

Consider the part of this expression which lies in $\Lambda^{0,1}(U) \otimes \Lambda^{1,0}(\mathbb{C}^k)$; it is:

$$\sum_{j=1}^{k} \left(\bar\partial F_{ij}(z,0) + \sum_{l=1}^{k} F_{il}(z,0) \tau_{lj} \right) \wedge dw_j$$

which means that

$$\bar\partial F_{ij}(z,0) + \sum_{l=1}^{k} F_{il}(z,0) \tau_{lj} = 0 \quad \forall i, j.$$

Therefore $f_{ij}(z) = F_{ij}(z,0)$ is a solution of (2.2.1). □

Example 2.2.6 (*Holomorphic structures on the trivial line bundle over an elliptic curve).* Let M be a compact 1-dimensional complex manifold diffeomorphic to the 2-dimensional torus $S^1 \times S^1$. Such a complex manifold is called an *elliptic curve* and arises as the quotient $\mathbb{C}/\Lambda$ by a lattice $\Lambda = \{m\omega_1 + n\omega_2;\, m, n \in \mathbb{Z}\}$, where ω_1, ω_2 are independent over $\mathbb{R}$. Consider the trivial complex line bundle $E = M \times \mathbb{C}$ over M. We want to consider possible holomorphic structures $\bar\partial$ on E. As observed in Remark 2.2.2, we only have to define $\bar\partial : \Gamma(E) \to \Omega^{0,1}(E)$. Since $\Gamma(E) \simeq C^\infty(M)$, its elements are Λ-periodic smooth functions on $\mathbb{C}$. On $\mathbb{C}$, a pseudoholomorphic structure is given simply by $\bar\partial f + B(z, \bar z) f d\bar z$, where $\bar\partial$ is the usual Dolbeault operator on $\mathbb{C}$ (i.e. $\bar\partial f = \frac{\partial f}{d\bar z} d\bar z$), and B is a smooth function.

Since this is supposed to define a pseudoholomorphic structure on M, B must be Λ-periodic. Since $\dim M = 1$, the condition $\bar{\partial}^2 = 0$ is trivially satisfied, and therefore any such B defines a holomorphic structure on $E \simeq M \times \mathbb{C}$ via:

$$\bar{\partial}_B f = \bar{\partial} f + B f d\bar{z}.$$

The question is when two such holomorphic structures are isomorphic, i.e. when are the corresponding holomorphic line bundles $L(B)$ isomorphic? Observe that $L_1 \simeq L_2$ is equivalent to $L_1 \otimes L_2^* \simeq \mathcal{O}_M$, where $\mathcal{O}_M$ denotes the trivial *holomorphic* line bundle on M. In addition, if $L_1 = L(B_1)$, $L_2 = L(B_2)$, then $L_1 \otimes L_2^* = L(B_1 - B_2)$. So we only need determine B such that $L(B)$ is holomorphically trivial. This is equivalent to $L(B)$ having a global holomorphic section which does not vanish anywhere (i.e. a global holomorphic frame). Therefore we want to determine all B such that there exists a Λ-periodic and nonvanishing smooth function f on $\mathbb{C}$, which satisfies $\bar{\partial} f + B f d\bar{z} = 0$.

First of all, I claim that if $B(0) = 0$, then such an f exists. Indeed, using Fourier series, we can then find a Λ-periodic function F such that $\frac{\partial F}{\partial \bar{z}} = B(z, \bar{z})$. The function $f = e^{-F}$ is then nonvanishing and $\bar{\partial}_B$-holomorphic, and, hence, $L(B) \simeq \mathcal{O}_M$.

Therefore we only need to consider the case $B = \text{const}$. The general solution to the equation $\bar{\partial} f + B f d\bar{z} = 0$ is then:

$$f(z, \bar{z}) = e^{-B\bar{z}} g(z),$$

where g is holomorphic. Since f is Λ-periodic, g satisfies

$$g(z + m\omega_1 + n\omega_2) = e^{B(m\overline{\omega}_1 + n\overline{\omega}_2)} g(z), \ \ \forall m, n \in \mathbb{Z}.$$

Moreover g is never zero, and hence, owing to the Weierstraß factorisation theorem, $g(z) = e^{h(z)}$ for an entire function $h(z)$, which satisfies

$$h(z + m\omega_1 + n\omega_2) = h(z) + B(m\overline{\omega}_1 + n\overline{\omega}_2) \mod 2\pi i\mathbb{Z}, \ \ \forall m, n \in \mathbb{Z}.$$

We may assume that $h(0) = 0$ (i.e. $f(0) = 1$). Then $h(z)/z$ is an entire function with bounded real part, hence constant. Therefore $h(z) = -Az$, $A \in \mathbb{C}$, and

$$m(A\omega_1 + B\overline{\omega}_1) + n(A\omega_2 + B\overline{\omega}_2) \in 2\pi i\mathbb{Z}, \ \ \forall m, n \in \mathbb{Z},$$

which means that $A\omega_1 + B\overline{\omega}_1 \in 2\pi i\mathbb{Z}$ and $A\omega_2 + B\overline{\omega}_2 \in 2\pi i\mathbb{Z}$. Solving this linear system finally gives

$$B = 2\pi i(\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2)^{-1}(k\omega_1 + l\omega_2),$$

for some $k, l \in \mathbb{Z}$. Thus we have shown that holomorphic line bundles on M, which are topologically trivial, are parametrised by $\mathbb{C}/\rho\Lambda$, where $\rho = 2\pi i(\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2)^{-1}$. Since rescaling the lattice corresponds to rescaling the coordinate z, $\mathbb{C}/\rho\Lambda \simeq \mathbb{C}/\Lambda$, i.e. these line bundles are parametrised by M itself.

Remark 2.2.7. Let E be a holomorphic vector bundle over a complex manifold M, and let $\bar{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ be the corresponding holomorphic structure. Since $\bar{\partial}^2 = 0$, we can define Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(M, E)$ of E in the usual way, i.e. as $\operatorname{Ker} \bar{\partial} / \operatorname{Im} \bar{\partial}$. Observe that if E is a trivial bundle of rank k, then $H^{p,q}_{\bar{\partial}}(M, E) \simeq H^{p,q}_{\bar{\partial}}(M) \otimes \mathbb{C}^k$. Observe also that $H^{p,0}_{\bar{\partial}}(M, E)$ is the vector space of global holomorphic E-valued p-forms; in particular, $H^{0,0}_{\bar{\partial}}(M, E)$ is the vector space of holomorphic sections of E. Next week we shall see a different approach to these cohomology groups.

Further reading:
Elliptic curves is a huge area and the literature is vast. My favourite introduction to the subject is: H. McKean and V. Moll, "Elliptic Curves: Function Theory, Geometry, Arithmetic" (CUP 1999).

2.3 Sheaves

Sheaf theory is an extremely useful technique for keeping track of local data and for passing (or identifying obstructions to passing) from local to global. It may seem somewhat abstract at the beginning, but it is, in fact, very natural. You actually know several sheaves and even use them: whenever you make an argument using an open neighborhood and continuous/differentiable/smooth maps on it, you are basically using an appropriate sheaf. The point of the theory is to extract the properties common to all such situations.

Let X be a topological space.

Definition 2.3.1. A *presheaf* $\mathcal{F}$ of (abelian) groups (resp. sets, rings, vector spaces etc.) consists of a group (resp. a set, ring, vector space etc.) $\mathcal{F}(U)$ for every open subset $U \subset X$, together with *restriction homomorphisms*

$$r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$$

for every inclusion of open sets $V \subset U$, which satisfy:

(1) r_{UU} is the identity on $\mathcal{F}(U)$;

(2) $r_{VW} \circ r_{UV} = r_{UW}$ for any $W \subset V \subset U$.

A basic example of a presheaf is the presheaf of continuous functions, i.e. $\mathcal{F}(U) = \{\text{continuous functions on } U\}$. Similarly, we have the presheaf of bounded continuous functions, and if X has aditional structure, e.g. smooth or holomorphic, we have presheaves of smooth or holomorphic functions. The restriction maps are exactly what the name says: they are restrictions of functions to a smaller subset. Observe that all of these are presheaves of (commutative) algebras.

Similarly, if E is a (topological) vector bundle on X, we have the presheaf of continuous sections of E. Because of the fundamental nature of this example, elements of $\mathcal{F}(U)$, for an arbitrary presheaf $\mathcal{F}$, are called *sections over* U. We shall write $s|_V$ for $r_{UV}(s)$.

Definition 2.3.2. A presheaf $\mathcal{F}$ is a *sheaf* if for every open set U and an open cover $\{U_i\}_{i\in I}$ of U the following two conditions hold:

(i) if $s,t\in\mathcal{F}(U)$ and $s|_{U_i}=t|_{U_i}$ $\forall i\in I$, then $s=t$;

(ii) if $s_i\in\mathcal{F}(U_i)$, $i\in I$, satisfy $s_i|_{U_i\cap U_j}=s_j|_{U_i\cap U_j}$ whenever $U_i\cap U_j\neq\emptyset$, then there exists an $s\in\mathcal{F}(U)$ such that $s|_{U_i}=s_i$ for all $i\in I$.

The first property says that a section is determined by its restrictions to arbitrarily small open subsets (locality). The second property means that we can glue local sections into a global one, as long as the obvious necessary condition is satisfied.

Examples 2.3.3. 1) The presheaf of continuous functions is a sheaf, denoted by C^0 (i.e. to each open U we attach $C^0(U)$). Similarly, on a smooth manifolds, we have sheaves C^k, $k\in\mathbb{N}$, and C^∞ of continuously differentiable or smooth functions. Observe that the presheaf of bounded continuous functions is not necessarily a sheaf: the gluing property will fail, unless X is compact.

2) Sheaves of locally constant $\mathbb{Z}$-, $\mathbb{R}$-, or $\mathbb{C}$-valued functions.

3) Sheaf Ω^p of smooth p-forms on a smooth manifold.

4) Sheaf $\Gamma(E)$ of smooth sections of a (real or complex) vector bundle on a manifold X.

Our main object of interest will be sheaves specific to complex manifolds:

$$\begin{aligned}\mathcal{O}&=\text{sheaf of holomorphic functions}\\ \mathcal{O}^*&=\text{sheaf of nowhere vanishing holomorphic functions}\\ \Omega^{p,q}&=\text{sheaf of forms of type }(p,q)\\ \mathcal{H}^{p,0}&=\text{sheaf of holomorphic }p\text{-forms}\end{aligned}$$

Observe that $\mathcal{O}$ is a sheaf of algebras, but $\mathcal{O}^*$ is only a sheaf of abelian groups (with respect to product of functions).

Morphism, kernels, cokernels, etc.

From now on we shall assume that our sheaves are always sheaves of (at least) abelian groups. This includes sheaves of vector spaces, commutative rings, etc.

Definition 2.3.4. A morphism $\alpha:\mathcal{F}\to\mathcal{G}$ between (pre-)sheaves on X consists of homomorphisms $\alpha_U:\mathcal{F}(U)\to\mathcal{G}(U)$ for all open subsets $U\subset X$ such that the following diagram commutes for all open inclusions $V\subset U$:

$$\begin{tikzcd}\mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "r_{UV}"] & \mathcal{G}(U) \arrow[d, "r_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V)\end{tikzcd}$$

Definition 2.3.5. The kernel of the morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is the sheaf $\mathrm{Ker}(\alpha)$ given by

$$\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}\big(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)\big).$$

It is easy to check that this assignment does, in fact, define a sheaf. The cokernel is harder. If we set, similarly,

$$\mathrm{Coker}(\alpha)(U) = \mathcal{G}(U)/\alpha_U(\mathcal{F}(U))$$

then we obtain a presheaf, but, as the following example shows, not necessarily a sheaf.

Example 2.3.6. Let $X = \mathbb{C}\backslash\{0\}$ and consider the morphism of sheaves $\exp : \mathcal{O} \to \mathcal{O}^*$, defined by $\mathcal{O}(U) \ni f \mapsto e^{2\pi i f} \in \mathcal{O}^*(U)$. The function $z \in \mathcal{O}^*(\mathbb{C}\backslash\{0\})$ is not in the image of exp, since one cannot define the logarithm on $\mathbb{C}\backslash\{0\}$, but its restriction to any contractible open set $U \subset \mathbb{C}\backslash\{0\}$ is in the image of $\mathcal{O}(U)$. Therefore z defines a nonzero element of $\mathrm{Coker}(\exp)(\mathbb{C}\backslash\{0\})$, but its restriction to every contractible U is 0 in $\mathrm{Coker}(\exp)(U)$, which contradicts property (i) of the definition of a sheaf.

Instead, we define an element of $\mathrm{Coker}(\alpha)(U)$ to be a collection $\{(U_i, s_i)\}$, where $\{U_i\}$ is an open cover of U and $s_i \in \mathcal{G}(U_i)$, such that

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \mathrm{Im}(\alpha_{U_i \cap U_j}) \text{ whenever } U_i \cap U_j \neq \emptyset.$$

We identify $\{(U_i, s_i)\}$ and $\{(U_i', s_i')\}$ if for any $p \in U_i \cap U_j'$ there exists an open set V with $p \in V \subset U_i \cap U_j'$ such that

$$s_i|_V - s_j'|_V \in \mathrm{Im}(\alpha_V).$$

Observe that with this definition, z in the above example is equal to 0 in $\mathrm{Coker}(\exp)(\mathbb{C}\backslash\{0\})$. We have *made* z satisfy condition (i) by localising it.

Remark 2.3.7. There is an analogous general procedure, called *sheafification* which turns any presheaf into a sheaf. Essentialy, it throws away sections which do not satisfy (i) and it adds sections which are missing in (ii).

Definition 2.3.8. A (short) sequence of sheaf morphisms

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

is *exact* if $\mathcal{E} = \mathrm{Ker}(\beta)$ and $\mathcal{G} = \mathrm{Coker}(\alpha)$.

We say then that $\mathcal{E}$ is a *subsheaf* of $\mathcal{F}$ and $\mathcal{G}$ is the *quotient sheaf* of $\mathcal{F}$ by $\mathcal{E}$, denoted $\mathcal{G} = \mathcal{F}/\mathcal{E}$. Observe that, given the definition of the cokernel sheaf, the condition $\mathcal{G} = \mathrm{Coker}(\alpha)$ means that for any section $s \in \mathcal{G}(U)$ and $p \in U$, there exists an open neighbourhood $V \subset U$ of p and a $t \in \mathcal{F}(V)$ such that $\beta_V(t) = s|_V$. In other words, any section of $\mathcal{G}$ is locally the image of a section of $\mathcal{F}$.

Example 2.3.9. On any complex manifold, the sequence

$$0 \to \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

is exact (here j is the inclusion and $\exp(f) = e^{2\pi i f}$). The meaning of exactness is: "given a nonvanishing holomorphic function, we can *locally* take its logarithm". This sequence is called the *exponential sheaf sequence.*

More generally, we say that a (long) sequence of sheaf morphisms

$$\dots \longrightarrow \mathcal{F}_n \xrightarrow{\alpha_n} \mathcal{F}_{n+1} \xrightarrow{\alpha_{n+1}} \mathcal{F}_{n+2} \longrightarrow \dots$$

is exact if $\alpha_{n+1} \circ \alpha_n = 0$ and

$$0 \to \mathrm{Ker}(\alpha_n) \xrightarrow{i} \mathcal{F}_n \xrightarrow{\alpha_n} \mathrm{Ker}(\alpha_{n+1}) \to 0$$

is exact for every n.

Example 2.3.10. 1) On any smooth manifold we have a sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \dots$$

of sheaves (real-valued functions and forms). It is exact. Indeed, $d^2 = 0$ and at every stage

$$0 \longrightarrow \mathrm{Ker}(d_n) \longrightarrow \Omega^n \xrightarrow{d_n} \mathrm{Ker}(d_{n+1}) \to 0$$

is exact, since the Poincaré lemma means that locally any form in $\mathrm{Ker}(d_{n+1})$ is in $\mathrm{Im}(d_n)$.

2) Similarly, on a complex manifold, the $\bar{\partial}$-Poincaré lemma implies that the sequence

$$0 \longrightarrow \mathcal{H}^{p,0} \xrightarrow{j} \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \Omega^{p,2} \xrightarrow{\bar{\partial}} \dots$$

is exact, where $\mathcal{H}^{p,0}$ denotes the sheaf of holomorphic p-forms.

Vector bundles and locally free sheaves

Let M be a smooth or complex manifold, and denote by $\mathcal{S}$ its *structure sheaf*, i.e. the sheaf C^∞ of smooth functions or the sheaf $\mathcal{O}$ of holomorphic functions. It is a sheaf of algebras. Let E be a vector bundle over M (in the respective category), and denote by $\mathcal{E}$ its sheaf of sections (respectively smooth or holomorphic). Then, for every open set U, $\mathcal{E}(U)$ is a module over $\mathcal{S}(U)$ (sections can be multiplied by functions). Moreover, if U is small enough so that $E|_U$ can be trivialised, then $\mathcal{E}(U) \simeq \mathcal{S}(U)^{\oplus k}$ ($k = \operatorname{rank} E$): any section can be written as $\sum f_i e_i$, where (e_i) is a local frame for E. In other words the module $\mathcal{E}(U)$ is free. One says that the sheaf of modules[7] $\mathcal{E}(U)$ is *locally free.*

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[7] A sheaf $\mathcal{M}$ is a *sheaf of modules* if $\mathcal{M}(U)$ is a module over $\mathcal{S}(U)$ for every open U, and the restriction maps r_{UV} for $\mathcal{S}$ and $\mathcal{M}$ are compatible, i.e. $r_{UV}(fm) = r_{UV}(f)\, r_{UV}(m)$.

Conversely, suppose that M is connected and that we are given a locally free sheaf of modules $\mathcal{E}(U)$ on M. This means that we have an open cover (U_α) of M and isomorphisms $g_\alpha : \mathcal{E}(U_\alpha) \simeq \mathcal{S}(U_\alpha)^{\oplus k}$. For any α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, we obtain (after restricting) an isomorphism

$$g_{\alpha\beta} = g_\alpha \circ g_\beta^{-1} : \mathcal{S}(U_\alpha \cap U_\beta)^{\oplus k} \longrightarrow \mathcal{S}(U_\alpha \cap U_\beta)^{\oplus k}.$$

This is nothing else than an invertible matrix of smooth or holomorphic functions on $U_\alpha \cap U_\beta$. These maps $g_{\alpha\beta}$ satisfy the compatibility conditions described at the beginning of §2.1 (p.27) and therefore define a vector bundle E on M.

It should be clear that these two constructions are inverse to each other (up to isomorphisms) and, consequently:

vector bundles = locally free sheaves of modules.

Example 2.3.11. Let M be a manifold and D a submanifold. Any vector bundle E on D can be extended, as a sheaf, to M, by setting $\tilde{\mathcal{E}}(U) = \mathcal{E}(U \cap D)$ if $U \cap D \neq \emptyset$, and $\tilde{\mathcal{E}}(U) = 0$ otherwise (with obvious restriction maps). This is a sheaf of modules on M which is not locally free.

Remark 2.3.12. The above equivalence between vector bundles and locally free sheaves is an equivalence of categories, so it is also an equivalence between morphisms. Be careful, however, about the meaning of an *injective morphism* under this equivalence. For example, consider the map χ from the trivial line bundle $M \times \mathbb{C}$ to itself, given by $(m, z) \mapsto (m, h(m)z)$, where h is a holomorphic function vanishing on $D \subsetneq M$. This is of course not injective on fibres over points of D, but it is a monomorphism in the category of vector bundles, i.e. if $g_1, g_2 : E \to M \times \mathbb{C}$ are two vector bundle morphisms from a vector bundle $\pi : E \to X$ such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. Observe that χ is clearly injective as a morphism of corresponding locally free sheaves (the product of a nonzero local section and a nonzero holomorphic function is nonzero).

Observe also that the cokernel of this monomorphism χ is the sheaf $\widetilde{\mathcal{O}}_D$, as introduced in the previous example, where $\mathcal{O}_D$ is the trivial bundle $D \times \mathbb{C}$ (assuming that D is a submanifold). Thus we see that the category of vector bundles (a.k.a. locally free sheaves) does not admit cokernels (nor kernels). In order to have those, one needs to enlarge the category to include the so-called *coherent* sheaves. These are those sheaves of $\mathcal{O}$-modules which arise locally as cokernels of morphisms between free sheaves.

Further reading:
Books devoted to sheaf theory tend to be very technical. It is better to read about sheaves in books on algebraic or complex analytic geometry. I recommend R. Wells' book from the literature list, or R. Vakil's online notes, available at
`http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`
If you are still sheaf-thirsty after that, then "Lectures on Algebraic Geometry I" by G. Harder (Springer 2011) is rather good.

2.4 Čech cohomology

In general, a cohomology theory identifies certain obstructions. The Čech cohomology identifies obstructions to patching local sections of a sheaf into a global one[8].

Let X be a topological space and $\mathcal{F}$ a sheaf (always of abelian groups) on X. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. We write

$$U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p},$$

and define

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha \in A} \mathcal{F}(U_\alpha) \\ C^1(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha \neq \beta \in A} \mathcal{F}(U_{\alpha\beta}) \\ &\vdots \\ C^p(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha_0 \neq \alpha_1 \neq \dots \neq \alpha_p \in A} \mathcal{F}(U_{\alpha_0 \alpha_1 \dots \alpha_p}). \end{aligned}$$

An element $s = (s_I) \in C^p(\mathcal{U}, \mathcal{F})$ is called a *p-cochain.* We define the *coboundary map*

$$\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

by

$$(\delta s)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_p} |_{U_{\alpha_0 \dots \alpha_{p+1}}}.$$

In particular if $s = \{s_U\} \in C^0(\mathcal{U}, \mathcal{F})$, then

$$(\delta s)_{UV} = (s_U)|_{U \cap V} - (s_V)|_{U \cap V}$$

and if $s = (s_{UV}) \in C^1(\mathcal{U}, \mathcal{F})$, then

$$(\delta s)_{UVW} = (s_{UV})|_{U \cap V \cap W} - (s_{UW})|_{U \cap V \cap W} + (s_{VW})|_{U \cap V \cap W}.$$

Lemma 2.4.1. *The composition $\delta \circ \delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+2}(\mathcal{U}, \mathcal{F})$ is the zero map.*

Proof.

$$\delta \circ \delta(s)_{\alpha_0 \dots \alpha_{p+2}} = \sum_{i,j} \underbrace{\left(\underbrace{(-1)^{j-1}(-1)^i}_{i \text{ deleted first}} + \underbrace{(-1)^i(-1)^j}_{j \text{ deleted first}} \right)}_{=0} s_{\alpha_0 \dots \hat{\alpha_i} \dots \hat{\alpha_j} \dots \alpha_{p+2}} = 0$$

□

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[8] This is a cop-out: already the first cohomology group does this. It is unclear to me what the geometric intuition behind higher Čech cohomology groups is.

A p-cochain $s \in C^p(\mathcal{U}, \mathcal{F})$ is called a *cocycle* if $\delta s = 0$, and a *coboundary* if $s = \delta t$ for some $t \in C^{p-1}(\mathcal{U}, \mathcal{F})$. We set

$$Z^p(\mathcal{U}, \mathcal{F}) = \operatorname{Ker}(\delta) \subset C^p(\mathcal{U}, \mathcal{F}) \quad \text{and} \quad \check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{Z^p(\mathcal{U}, \mathcal{F})}{\delta(C^{p-1}(\mathcal{U}, \mathcal{F}))}.$$

Thus $\check{H}^p(\mathcal{U}, \mathcal{F})$ is the p-th cohomology group of the complex

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \dots, \tag{2.4.1}$$

and it depends on the choice of an open cover $\mathcal{U}$.

Example 2.4.2. Let $X = \mathbb{P}^1$ and $\mathcal{F} = \mathcal{O}$. We have the open cover $U_0 = \{[z_0, z_1] \mid z_0 \neq 0\}$, $U_1 = \{[z_0, z_1] \mid z_1 \neq 0\}$. Both are isomorphic to $\mathbb{C}$ and the intersection is $\mathbb{C}^*$. The map

$$\begin{gathered} \delta : \; C^0(\mathcal{U}, \mathcal{F}) = \mathcal{O}(U) \oplus \mathcal{O}(V) \to C^1(\mathcal{U}, \mathcal{F}) = \mathcal{O}(U \cap V) \\ \delta(f, g) = f(z) - g\left(\frac{1}{z}\right). \end{gathered}$$

We can expand f as a power series in z, g as a power series in $\frac{1}{z}$, and hence

$$\delta(f, g) = 0 \Longleftrightarrow f = g = \text{const.}$$

Therefore $\check{H}^0(\mathcal{U}, \mathcal{O}) = \mathbb{C}$. Now observe that the image of δ consists of all holomorphic functions on $\mathbb{C}^*$: take a Laurent series expansion and set

$$\begin{aligned} f(z) &= \text{nonnegative powers of } z \\ -g\left(\frac{1}{z}\right) &= \text{negative powers of } z. \end{aligned}$$

Hence $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$ (and, of course, that is all, since the complex (2.4.1) terminates at $p = 1$).

Now recall that, given two covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{U}' = \{U'_\beta\}_{\beta \in B}$ of X, we say that $\mathcal{U}'$ is a *refinement* of $\mathcal{U}$ if for every $\beta \in B$ there exists $\alpha \in A$ such that $U'_\beta \subset U_\alpha$. We write then $\mathcal{U}' \leq \mathcal{U}$. For each β choose α as above and denote it by $\varphi(\beta)$; this defines a function $\varphi : B \to A$. We then obtain a map

$$\rho_\varphi : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}', \mathcal{F}), \qquad \rho_\varphi(s)_{\beta_0 \dots \beta_p} = s_{\varphi(\beta_0) \dots \varphi(\beta_p)}|_{U'_{\beta_0 \dots \beta_p}}. \tag{2.4.2}$$

This commutes with δ, and therefore induces a map on cohomology

$$\rho_{\mathcal{U}\mathcal{U}'} = \rho_\varphi : \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{U}', \mathcal{F}).$$

One can check that this does not depend on the choice of φ (the maps ρ_ϕ and ρ_ψ for two such choices are *chain-homotopic* and therefore induce the same map on cohomology). We define the *p-th Čech cohomology* of $\mathcal{F}$ on X as the direct limit[9] of the $\check{H}^p(\mathcal{U},\mathcal{F})$ as $\mathcal{U}$ becomes finer and finer:

$$\check{H}^p(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U},\mathcal{F}).$$

This definition is clearly impossible to work with in practice. We need a simple sufficient condition on a cover $\mathcal{U}$ so that

$$\check{H}(\mathcal{U},\mathcal{F}) = \check{H}(X,\mathcal{F}).$$

In other words the direct limit stabilises at $\mathcal{U}$, and further refinements do not change anything. Such a condition is provided by

Theorem 2.4.3 (Leray's Theorem)**.** *If a cover $\mathcal{U}$ is acyclic for a sheaf $\mathcal{F}$ in the sense that*

$$\check{H}^p(U_{i_1}\cap\cdots\cap U_{i_q},\mathcal{F}) = 0 \quad \forall p>0 \;\; \forall i_1,\ldots,i_q,$$

then $\check{H}^*(\mathcal{U},\mathcal{F}) = \check{H}^*(X,\mathcal{F})$.

Such a cover is also called a *Leray cover*. We shall not prove Leray's theorem in full generality, only in those cases where it will be used.

Remark 2.4.4. Observe, directly from the definition, that $\check{H}^0(X,\mathcal{F}) = \check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X)$ for any open cover $\mathcal{U}$, i.e. the 0-th Čech cohomology group is the space of *global sections* of $\mathcal{F}$. This justifies our notation $H^0(M,E)$ for the space of holomorphic sections of a holomorphic vector bundle.

Remark 2.4.5. The correct definition of *sheaf cohomology* uses homological algebra and is very difficult to use in computations. Fortunately, Čech cohomology is isomorphic to sheaf cohomology for paracompact[10] spaces.

We shall now introduce the main computational tool in cohomology: the long exact sequence. Let $f:\mathcal{F}\to\mathcal{G}$ be a morphism of sheaves. This induces a map $C^p(\mathcal{U},\mathcal{F})\to C^p(\mathcal{U},\mathcal{G})$ for any open cover $\mathcal{U}$, which commutes with δ, and therefore induces a map on cohomology

$$f_*:\check{H}^p(X,\mathcal{F}) \longrightarrow \check{H}^p(X,\mathcal{G}) \quad \forall p.$$

A fundamental property of sheaf cohomology is:

Theorem 2.4.6. *Suppose that*

$$0\to\mathcal{E}\xrightarrow{f}\mathcal{F}\xrightarrow{g}\mathcal{G}\to 0$$

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[9] The direct limit is defined as the quotient of the direct sum over all open covers by an equivalence relation, where $x\in\check{H}^p(\mathcal{U}_1,\mathcal{F})$ and $y\in\check{H}^p(\mathcal{U}_2,\mathcal{F})$ are equivalent, if there exists a common refinement U' such that $\rho_{\mathcal{U}_1\mathcal{U}'}(x)=\rho_{\mathcal{U}_2\mathcal{U}'}(y)$.

[10] I.e. a Hausdorff topological space such that any open cover admits a locally finite refinement.

is a short exact sequence of sheaves on a paracompact space X. Then there exist natural maps

$$\delta^* : \check{H}^p(X, \mathcal{G}) \longrightarrow \check{H}^{p+1}(X, \mathcal{E})$$

such that the following long sequence on cohomology is exact:

$$\cdots \longrightarrow \check{H}^{p-1}(X, \mathcal{G}) \xrightarrow{\delta^*} \check{H}^p(X, \mathcal{E}) \xrightarrow{f_*} \check{H}^p(X, \mathcal{F}) \xrightarrow{g_*} \check{H}^p(X, \mathcal{G}) \xrightarrow{\delta^*} \cdots.$$

Proof. The idea behind long exact cohomology sequences is always the same and it involves "diagram chasing" or the "zig-zag lemma" (recall the proof of exactness of the Mayer-Vietoris sequence in the de Rham cohomology), provided we are dealing with a short exact sequence of chain complexes of *abelian groups*. Here the problem is that the exactness of a short sequence of sheaves does not imply exactness of

$$0 \to \mathcal{E}(U) \xrightarrow{f} \mathcal{F}(U) \xrightarrow{g} \mathcal{G}(U) \to 0$$

for an open subset U (recall Example 2.3.6). We need to adapt the arguments to this situation.

We begin by constructing δ^*. Suppose that $s \in \check{H}^p(X, \mathcal{G})$ is represented by a cocycle $s \in C^p(\mathcal{U}, \mathcal{G})$. We may assume that $\mathcal{U} = \{U_j\}_{j \in J}$ is locally finite. We can then find a cover $\mathcal{V} = \{V_j\}_{j \in J}$ such that $\overline{V}_j \subset U_j$ for all $j \in J$. For every $x \in X$ we choose an open neighbourhood W_x, so that the following conditions are satisfied:

(i) if $x \in U_{j_0 \ldots j_p}$, then there exists a section $t \in \mathcal{F}(W_x)$ such that $g(t) = s_{j_0 \ldots j_p}|_{W_x}$;

(ii) if $x \in U_j$ (resp. $x \in V_j$), then $W_x \subset U_j$ (resp. $W_x \subset V_j$);

(iii) if $W_x \cap V_j \neq \emptyset$, then $W_x \subset U_j$.

Existence of a W_x satisfying (i) follows from the definition of the cokernel of a sheaf morphism. We can then ensure (ii) and (iii) for x in some $U_{j_0 \ldots j_p}$ by making W_x smaller (since $\mathcal{U}$ is locally finite, x belongs to only finitely many $U_{j_0 \ldots j_p}$). For x which do not belong to any $U_{j_0 \ldots j_p}$, we only need to ensure (ii) and (iii) which is easy to do.

The family $\mathcal{W} = \{W_x\}_{x \in X}$ is an open cover of X, and for every x we can find a $\varphi(x) \in J$ such that $W_x \subset V_j$. We consider $\rho_\varphi(s) \in C^p(\mathcal{W}, \mathcal{G})$, where ρ_φ is the map defined in (2.4.2). I claim that there exists $t \in C^p(\mathcal{W}, \mathcal{F})$ such that $\rho_\varphi(s) = g(t)$. Consider $W_{x_0} \cap \cdots \cap W_{x_p}$. If it is empty, there is nothing to show. If not, then $W_{x_0} \cap W_{x_i} \neq \emptyset$ for $i = 1, \ldots, p$, and since $W_{x_i} \subset V_{\varphi(x_i)}$, condition (iii) above implies that $W_{x_0} \subset U_{\varphi(x_i)}$ for $i = 1, \ldots, p$. Therefore $x_0 \in U_{\varphi(x_0) \ldots \varphi(x_p)}$. Property (i) guarantees that there exists a section $t \in \mathcal{F}(W_{x_0})$ such that $g(t) = s_{\varphi(x_0) \ldots \varphi(x_p)}$ on W_{x_0}, and therefore also on $W_{x_0} \cap \cdots \cap W_{x_p}$.

We have shown that there exists a refinement $\mathcal{W} \leq \mathcal{U}$ such that $\rho_\varphi(s) = g(t)$ for some $t \in C^p(\mathcal{W}, \mathcal{F})$. But then

$$g(\delta t) = \delta g(t) = \delta \rho_\varphi(s) = \rho_\varphi(\delta s) = 0.$$

Exactness in the middle of the short exact sequence implies now that there is a $u \in C^{p+1}(\mathcal{W}, \mathcal{E})$ such that $f(u) = \delta t$. Then

$$f(\delta u) = \delta(f(u)) = \delta^2 t = 0.$$

Since f is injective, $\delta u = 0$ and we can define $\delta^*(s) = [u] \in \check{H}^{p+1}(\mathcal{W}, \mathcal{E})$. Passing to direct limits defines $\delta^*(s) \in \check{H}^{p+1}(X, \mathcal{E})$.

The proof of exactness of the long sequence follows now the usual lines (as for the Mayer-Vietoris sequence), as long as we use the existence of a refinement $\mathcal{W} \leq \mathcal{U}$ as above. □

The following corollary is useful:

Corollary 2.4.7. *Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of sheaves on a topological space X. If $U \subset X$ is a paracompact open subset such that $\check{H}^1(U, \mathcal{E}) = 0$, then $0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0$ is a short exact sequence of abelian groups.*

Proof. Apply the above theorem to $X = U$ and use the fact that $H^0(U, \mathcal{E}) = \mathcal{E}(U)$ etc. □

In general, the following types of sheaves are of interest on a manifold:

1) locally constant sheaves $\mathbb{Z}, \mathbb{R}, \mathbb{C}$. These carry topological information. We shall see shortly that if M is a smooth manifold, then

$$\check{H}^*(M, \mathbb{R}) = H^*_{dR}(M).$$

2) C^∞-sheaves such as the sheaf of smooth functions, Ω^p, or $\Omega^{p,q}$ on an almost complex manifold. Their local sections can be expressed locally as n-tuples of C^∞-functions. Their cohomology is trivial (see below).

3) holomorphic sheaves such as $\mathcal{O}$, $\mathcal{H}^{p,0}$ (holomorphic p-forms), sheaf of holomorphic sections of a vector bundle. For these, the Čech cohomology carries a lot of information.

Let us prove the statement made in 2).

Proposition 2.4.8. *Let M be an almost complex manifold. Then*

$$\check{H}^r(M, \Omega^{p,q}) = 0 \quad \forall r > 0.$$

Proof. Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a locally finite cover with a subordinate partition of unity $(\lambda_\alpha)_{\alpha \in A}$. For an $s \in Z^r(\mathcal{U}, \Omega^{p,q})$ define $t \in C^{r-1}(\mathcal{U}, \Omega^{p,q})$ by

$$t_{\alpha_0 \ldots \alpha_{r-1}} = \sum_{\beta \in A} \lambda_\beta s_{\beta \alpha_0 \ldots \alpha_{r-1}}.$$

This is well-defined: although $s_{\beta\alpha_0\dots\alpha_{r-1}}$ is defined only on $U_\beta \cap U_{\alpha_0} \cap \dots \cap U_{\alpha_{r-1}}$, we can extend $\lambda_\beta s_{\beta\alpha_0\dots\alpha_{r-1}}$ to $U_{\alpha_0\dots\alpha_{r-1}}$ by zero, since $\operatorname{supp}(\lambda_\beta) \subset U_\beta$.
One checks then that $\delta t = s$. E.g. for $r = 1$, we have $s = (s_{UV})$ and $\delta s = 0$ means that $s_{UV} - s_{UW} + s_{VW} = 0$ on $U \cap V \cap W$. Then $t_U = \sum\limits_V \lambda_V s_{VU}$ and

$$(\delta t)_{UV} = t_U - t_V = \sum_W \lambda_W s_{WU} - \sum_W \lambda_W s_{WV} = \sum_W \lambda_W s_{UV} \underset{\Sigma\lambda=1}{=} s_{UV}.$$

The argument for higher r is completely analogous, if notationally more complicated. □

Remark 2.4.9. Sheaves admitting partitions of unity are called *fine*, and the same argument shows that their higher cohomology groups vanish.

Theorem 2.4.10 (Dolbeault theorem)**.** *Let M be a complex manifold, and $\mathcal{H}^{p,0}$ the sheaf of holomorphic p-forms on M. Then*

$$\check{H}^q(M, \mathcal{H}^{p,0}) = H^{p,q}_{\bar\partial}(M) \quad \forall q.$$

Proof. As observed in the previous section, the $\bar\partial$-Poincaré lemma implies that the sequences

$$0 \to \mathcal{H}^{p,0} \xrightarrow{\bar\partial} \Omega^{p,0} \xrightarrow{\bar\partial} Z^{p,1}_{\bar\partial} \to 0,$$

$$\vdots$$

$$0 \to Z^{p,q}_{\bar\partial} \xrightarrow{\bar\partial} \Omega^{p,q} \xrightarrow{\bar\partial} Z^{p,q+1}_{\bar\partial} \to 0,$$

are exact for all q. The associated long exact sequences are:

$$\dots \longrightarrow \check{H}^r(\Omega^{p,0}) \xrightarrow{\bar\partial_*} \check{H}^r(Z^{p,1}_{\bar\partial}) \xrightarrow{\delta^*} \check{H}^{r+1}(\mathcal{H}^{p,0}) \xrightarrow{\bar\partial_*} \check{H}^{r+1}(\Omega^{p,0}) \longrightarrow \dots$$

$$\vdots$$

$$\dots \longrightarrow \check{H}^r(\Omega^{p,q}) \xrightarrow{\bar\partial_*} \check{H}^r(Z^{p,q+1}_{\bar\partial}) \xrightarrow{\delta^*} \check{H}^{r+1}(Z^{p,q}_{\bar\partial}) \xrightarrow{\bar\partial_*} \check{H}^{r+1}(\Omega^{p,q}) \longrightarrow \dots.$$

Since $\check{H}^r(\Omega^{p,q}) = 0 \ \ \forall r > 0$, we obtain

$$\begin{aligned}\check{H}^r(\mathcal{H}^{p,0}) &\simeq \check{H}^{r-1}(Z^{p,1}_{\bar\partial}) \simeq \check{H}^{r-2}(Z^{p,2}_{\bar\partial}) \simeq \dots \\ &\simeq \check{H}^1(Z^{p,r-1}_{\bar\partial}) \simeq \check{H}^0(Z^{p,r}_{\bar\partial})/\bar\partial_*(\check{H}^0(\Omega^{p,r-1})) = H^{p,r}_{\bar\partial}(M),\end{aligned}$$

where the last equality follows from the fact that $\check{H}^0$ is the space of global sections of the given sheaf. □

Remark 2.4.11. The same argument applied to the first exact sequence in Example 2.3.10 proves the *de Rham theorem*: $\check{H}^*(M, \mathbb{R}) = H^*_{\mathrm{dR}}(M)$ on any smooth manifold.

In Remark 2.2.7 we introduced Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(M,E)$ of a holomorphic vector bundle. These have the following sheaf-theoretic interpretation:

Proposition 2.4.12. *Let $\pi : E \to M$ be a holomorphic vector bundle on a complex manifold, and denote by $\mathcal{H}^{p,0}(E)$ the sheaf of E-valued holomorphic p-forms on M. Then:*

$$H^{p,q}_{\bar\partial}(M,E) = \check{H}^q(M,\mathcal{H}^{p,0}(E)) \quad \forall q.$$

Proof. Identical to the proof of Dolbeault's theorem. □

As an application of Dolbeault's theorem we shall prove the Leray theorem for the sheaf $\mathcal{O}$ of holomorphic functions:

Proposition 2.4.13. *If $\{U_\alpha\}$ is a locally finite cover, which is acyclic for $\mathcal{O}$, then*

$$\check{H}^p(\mathcal{U},\mathcal{O}) \simeq \check{H}^p(M,\mathcal{O}) \quad \forall p.$$

Proof. From the Dolbeault theorem:

$$H^{0,r}_{\bar\partial}(U_{\alpha_1}\cap\cdots\cap U_{\alpha_s}) = \check{H}^r(U_{\alpha_1}\cap\cdots\cap U_{\alpha_s},\mathcal{O}) = 0.$$

This means that that we have a short exact sequence

$$0 \to Z^{0,r-1}_{\bar\partial}(U_{\alpha_1\ldots\alpha_s}) \xrightarrow{j} \Omega^{0,r-1}(U_{\alpha_1\ldots\alpha_s}) \xrightarrow{\bar\partial} Z^{0,r}_{\bar\partial}(U_{\alpha_1\ldots\alpha_s}) \to 0.$$

Since, by assumption, this is true for all multi-intersections, the definition of a sheaf implies that the exactness holds at the level of cochains:

$$0 \to C^s(\mathcal{U},Z^{0,r-1}_{\bar\partial}) \longrightarrow C^s(\mathcal{U},\Omega^{0,r-1}) \longrightarrow C^s(\mathcal{U},Z^{0,r}_{\bar\partial}) \to 0.$$

In the associated long exact cohomology sequence the middle terms vanish for all $s>0$, by a partition of unity argument. Therefore

$$\check{H}^s(\mathcal{U},Z^{0,r}_{\bar\partial}) \simeq \check{H}^{s+1}(\mathcal{U},Z^{0,r-1}_{\bar\partial}) \quad \forall r\geq 0, s>0.$$

It follows:

$$\begin{aligned}\check{H}^p(\mathcal{U},\mathcal{O}) &\simeq \check{H}^{p-1}(\mathcal{U},Z^{0,1}_{\bar\partial}) \simeq \check{H}^{p-2}(\mathcal{U},Z^{0,2}_{\bar\partial}) \simeq \cdots \simeq \check{H}^1(\mathcal{U},Z^{0,p-1}_{\bar\partial})\\ &\simeq H^0(\mathcal{U},Z^{0,p}_{\bar\partial})/\bar\partial(H^0(\mathcal{U},\Omega^{0,p-1})) = H^{0,p}_{\bar\partial}(M) = \check{H}^p(M,\mathcal{O}),\end{aligned}$$

where the last equality is again due to Dolbeault's theorem. □

Remark 2.4.14. The same argument works for the sheaves $\mathcal{H}^{p,0}$ of holomorphic p-forms and the sheaves $\mathcal{H}^{p,0}(E)$ of holomorphic E-valued p-forms.

Further reading:
At the end of §1.6 I mentioned the so-called *Stein manifolds*, as an example of a class of manifolds with trivial Dolbeault cohomology. This follows from a deep theorem, Cartan's theorem B, which states that on a Stein manifold (and even on a *Stein space*) $\check{H}^p(M, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ and all $p > 0$. The proof of this is really very complicated, see H. Grauert and R. Remmert, "Theory of Stein spaces" (Springer 1979).
It is perhaps of interest that the following natural question appears not to have been answered yet (at least I could not find an answer): does the vanishing of $H^{p,q}_{\bar{\partial}}(M)$ for all $p \geq 0$ and $q > 0$ imply that M is Stein?

Chapter 3

Connections, curvature, metrics

3.1 Connections and their curvature

Let $\pi : E \to M$ be a complex vector bundle on a smooth manifold M. Sections of E form a vector space and can, in many ways, be viewed as a generalisation of smooth functions (which are sections of the trivial bundle $M \times \mathbb{C}$). There is, however, an important difference: there is no canonical way to differentiate sections, i.e. no linear operator $\Gamma(E) \to \Gamma(E)$ which behaves locally like a first order differential operator. We have to introduce such an operator per hand:

Definition 3.1.1. A *connection* on a complex vector bundle $E \xrightarrow{\pi} M$ (over a smooth manifold M) is a linear map

$$D : \Gamma(E) \to \Omega^1(E)$$

which satisfies the Leibniz rule

$$D(fs) = df \otimes s + f Ds \quad \forall\, f \in C^\infty(M),\ \forall\, s \in \Gamma(E).$$

Observe that for each tangent vector $v \in T_xM$ we obtain an operator $D_v : \Gamma(E) \to E_x$, $D_v(s) = (Ds)(v)$ (evaluation of a 1-form on a tangent vector), which should be viewed as analogous to directional derivative.

If we choose a local frame $e = (e_1, \dots, e_k)$ for E over U, then we can write

$$De_i = \sum_{j=1}^{k} \vartheta_{ij} \otimes e_j$$

for a matrix $\vartheta_e = [\vartheta_{ij}]$ of 1-forms, called the *connection matrix* (with respect to the frame e). The data e and ϑ_e determine D: for a general section $s = \sum\limits_{i=1}^{k} f_i e_i$

we obtain

$$Ds = \sum_{i=1}^{k} df_i \otimes e_i + \sum_{i=1}^{k} f_i De_i = \sum_{j=1}^{k} \left(df_j + \sum_{i=1}^{k} f_i \vartheta_{ij} \right) \otimes e_j.$$

If $e' = (e'_1, \dots, e'_k)$ is another local frame with $e'(z) = g(z)e(z)$, for a transition function $z \mapsto g(z) \in GL(k, \mathbb{C})$, then

$$De'_i = D\left(\sum_{j=1}^{k} g_{ij} e_j\right) = \sum_{j=1}^{k} dg_{ij} \otimes e_j + \sum_{j=1}^{k} g_{ij} De_j = \sum_{j=1}^{k} dg_{ij} \otimes e_j + \sum_{j,k=1}^{k} g_{ik} \vartheta_{kj} \otimes e_j$$
$$= \sum_{j,l=1}^{k} (dg_{ij} + g_{il}\vartheta_{lj}) \otimes e_j = \sum_{j,l,s=1}^{k} (dg_{ij} + g_{il}\vartheta_{lj}) \otimes \left(g^{-1}\right)_{js} e'_s.$$

In other words, the transformation law for the connection matrix is:

$$\vartheta_{e'} = g\vartheta_e g^{-1} + dg g^{-1}.$$

Remark 3.1.2. A connection on a vector bundle $E \xrightarrow{\pi} M$ induces a connection on any vector bundle which can be obtained from E by linear operations, e.g. E^*, $\Lambda^p E$, $\mathrm{Hom}(E, E)$, etc. Similarly, given connections on vector bundles $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$, we obtain a canonical connection on $E_1 \oplus E_2$, $E_1 \otimes E_2$, etc. I shall leave the details as an exercise (Homework 6).

Remark 3.1.3. Every vector bundle admits a connection by a partition of unity argument - see Proposition 3.2.5 below. The space of connections on E is acted upon by the *gauge group*, i.e. automorphisms of E which preserve fibres. If g is such an automorphism, then the action is defined by $D \mapsto D^g$, where $D^g(s) = gD(g^{-1}s)$.

Curvature

We can extend any connection to act on $\Omega^p(E)$, $p \geq 1$, by imposing the Leibniz rule

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge Ds, \quad \forall\, \omega \in \Omega^p(M), \ \forall\, s \in \Gamma(E).$$

In particular we can consider the operator

$$D^2 = D \circ D : \Gamma(E) \to \Omega^2(E).$$

Unlike D, D^2 is linear over functions:

Proposition 3.1.4. *D^2 is linear over $C^\infty(M)$, i.e.*

$$D^2(fs) = fD^2(s) \quad \forall\, f \in C^\infty(M), \ \forall\, s \in \Gamma(E).$$

Proof.

$$D^2(fs) = D(df \otimes s + fDs) = \underbrace{-df \wedge Ds + df \wedge Ds}_{=0} + fD^2 s = fD^2 s.$$

□

This means that the value $D^2(s)(x)$ at a point $x \in M$ depends only on $s(x)$, and not on the first derivatives of s. In other words D^2 is induced by a bundle map $E \to \Lambda^2 M \otimes E$, i.e. a section of $\Lambda^2 M \otimes \mathrm{Hom}(E, E)$, which is the same as a $\mathrm{Hom}(E, E)$-valued 2-form R^D. R^D is called the *curvature* of the connection.

If $e = (e_1, \ldots, e_k)$ is a local frame for E, then we can represent $R^D = D^2$ by a matrix of 2-forms

$$D^2 e_i = \sum_{j=1}^{k} \Theta_{ij} \otimes e_j.$$

The matrix $\Theta_e = [\Theta_{ij}]$ is called the *curvature matrix* with respect to the frame e. If $e'_i = \sum\limits_{j=1}^{k} g_{ij} e_j$ is another frame, then

$$D^2 e'_i = D^2 \left(\sum_{j=1}^{k} g_{ij} e_j \right) = \sum_{j=1}^{k} g_{ij} D^2 e_j = \sum_{j,l=1}^{k} g_{ij} \Theta_{jl} \otimes e_l =$$

$$= \sum_{j,l,s=1}^{k} g_{ij} \Theta_{jl} \otimes (g^{-1})_{ls} e'_s = \sum_{j,l,s=1}^{k} \big(g_{ij} \Theta_{jl} (g^{-1})_{ls} \big) \otimes e'_s,$$

so that

$$\Theta_{e'} = g \Theta_e g^{-1}.$$

We can express the curvature matrix in terms of the connection matrix: since $De_i = \sum\limits_{j=1}^{k} \vartheta_{ij} e_j$, we obtain

$$D^2 e_i = D \left(\sum_{j=1}^{k} \vartheta_{ij} e_j \right) = \sum_{j=1}^{k} \left(d\vartheta_{ij} - \sum_{p=1}^{k} \vartheta_{ip} \wedge \vartheta_{pj} \right) \otimes e_j.$$

We can write this as

$$\Theta_e = d\vartheta_e - \vartheta_e \wedge \vartheta_e, \tag{3.1.1}$$

where $\wedge$ denotes matrix product with respect to the wedge product (i.e. exactly what the formula above says). Equations (3.1.1) are called *Cartan structure equations.*

3.2 Hermitian metrics

Definition 3.2.1. Let $E \overset{\pi}{\to} M$ be a complex vector bundle. A *hermitian metric* on E is a smoothly varying hermitian inner product on each fibre E_x (i.e. if $e = (e_1, \ldots, e_k)$ is a smooth frame for E, then the functions $h_{ij}(x) = \langle e_i(e), e_j(x)\rangle$ are C^∞). A complex vector bundle equipped with a hermitian metric is called a *hermitian vector bundle.*

Example 3.2.2. Recall the tautological bundle $J_{\mathbb{CP}^n}$ on $\mathbb{CP}^n$. Its fibre over $z \in \mathbb{CP}^n$ is just the line $\langle z\rangle$ in $\mathbb{C}^{n+1}$. We can define a hermitian metric on J by simply restricting the standard hermitian inner product on $\mathbb{C}^{n+1}$ to $\langle z\rangle$.

Definition 3.2.3. Let $E \overset{\pi}{\to} M$ be a hermitian vector bundle. A connection $D : \Gamma(E) \to \Omega^1(E)$ is called *hermitian* if it is compatible with the metric, i.e.

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle \quad \forall\, s_1, s_2 \in \Gamma(E).$$

Another way of saying this is that the metric is parallel with respect to D - see Homework 6 for details. If $e = (e_1, \ldots, e_k)$ is a local frame for E and we put $h_{ij} = \langle e_i, e_j\rangle$, then this condition reads

$$dh_{ij} = \langle De_i, e_j\rangle + \langle e_i, De_j\rangle = \sum_{p=1}^{k} \vartheta_{ip} h_{pj} + \sum_{p=1}^{k} \overline{\vartheta}_{jp} h_{ip} \quad \forall\, i, j. \tag{3.2.1}$$

Remark 3.2.4. In a unitary trivialisation (cf. Homework 6), it follows from (3.2.1) that the connection matrix is skew-hermitian, i.e. $\overline{\vartheta}_{ij} = -\vartheta_{ji}$ for all i, j. The same holds then for the curvature matrix.

Proposition 3.2.5. *Any vector bundle admits a hermitian metric. Any hermitian vector bundle admits a compatible connection.*

Proof. Let $\{U_i\}_{i\in I}$ be a locally finite cover such that each $E|_{U_i}$ is trivial, and let $\{\lambda_i\}$ be a subordinate partition of unity. On each $E|_{U_i}$ there is a hermitian metric h_i and we set $h = \sum_{i\in I} \lambda_i h_i$. For the second statement, let $\langle\ ,\ \rangle$ be a hermitian metric on E. On each $E|_{U_i}$ we can find a hermitian connection D_i (for example with trivial connection matrix in a unitary trivialisation). Then

$$Ds = \sum_{i\in I} D_i(\lambda_i s) = \sum_{i\in I} \lambda_i D_i s,$$

is a connection on E and we check that it is hermitian:

$$\begin{aligned}\langle Ds, t\rangle + \langle s, Dt\rangle &= \Big\langle \sum_{i\in I} \lambda_i D_i s, t \Big\rangle + \Big\langle s, \sum_{i\in I} \lambda_i D_i t \Big\rangle = \\ &= \sum_{i\in I} \lambda_i \Big(\big\langle D_i s, t\big\rangle + \big\langle s, D_i t\big\rangle \Big) = \sum_{i\in I} \lambda_i d\big\langle s, t\big\rangle = d\langle s, t\rangle.\end{aligned}$$

□

Suppose now that M is a complex manifold and E is a holomorphic vector bundle. We can decompose a connection $D : \Gamma(E) \to \Omega^1(E)$ as $D = D^{1,0} + D^{0,1}$, where

$$D^{1,0} : \Gamma(E) \to \Omega^{1,0}(E) \quad \text{and} \quad D^{0,1} : \Gamma(E) \to \Omega^{0,1}(E).$$

This much is true on any complex vector bundle over an almost complex manifold. However, on a holomorphic E we already have a differential operator $\bar\partial : \Gamma(E) \to \Omega^{0,1}(E)$ - the natural holomorphic structure defined at the beginning of §2.2. We therefore call a connection D *compatible with the complex structure* if $D^{0,1} = \bar\partial$.

Theorem 3.2.6. *If $E \xrightarrow{\pi} M$ is a hermitian holomorphic vector bundle, then there exists a unique connection D (called the* Chern connection*) compatible with both the metric and the complex structure.*

Proof. Let $e = (e_1, \ldots, e_k)$ be a local holomorphic frame for E and put $h_{ij} = \langle e_i, e_j \rangle$. If D is compatible with the complex structure, then De_i is of type $(1, 0)$ for each i. Let ϑ be the connection matrix of D with respect to e, i.e. $De_i = \sum_{j=1}^{k} \vartheta_{ij} \otimes e_j$. It follows that ϑ_{ij} are of type $(1, 0)$.

On the other hand, if D is compatible with the metric, then we have equation (3.2.1). Hence, if D is compatible with both complex structure and the metric, then, after decomposing according to type,

$$\partial h_{ij} = \sum_{p=1}^{k} \vartheta_{ip} h_{pj}, \quad \bar\partial h_{ij} = \sum_{p=1}^{k} \overline{\vartheta}_{jp} h_{ip},$$

or in matrix notation $\partial h = \vartheta h$, $\bar\partial h = h\vartheta^*$. Now just observe that $\vartheta = (\partial h)h^{-1}$ is a unique solution to both equations. □

Let us discuss the curvature of a Chern connection. Recall formula (3.1.1) for curvature matrix of a connection D with respect to a frame e:

$$\Theta_e = d\vartheta_e - \vartheta_e \wedge \vartheta_e = d\vartheta_e - [\vartheta_e, \vartheta_e].$$

If D is the Chern connection on a holomorphic hermitian vector bundle and e is a holomorphic frame, then we have just seen that

$$\vartheta_e = \partial h h^{-1}, \quad \text{where} \quad h_{ij} = \langle e_i, e_j \rangle.$$

We now compute:

$$\begin{aligned} d\vartheta_e &= (\partial + \bar\partial)\vartheta_e = \bar\partial\vartheta_e + \partial(\partial h h^{-1}) = \bar\partial\vartheta_e - \partial h \wedge (\partial h^{-1}) \\ &= \bar\partial\vartheta_e + \partial h \wedge h^{-1}\partial h h^{-1} = \bar\partial\vartheta_e + \partial h h^{-1} \wedge \partial h h^{-1}. \end{aligned}$$

Hence

$$\Theta_e = d\vartheta_e - \vartheta_e \wedge \vartheta_e = \bar\partial\vartheta_e = \bar\partial(\partial h h^{-1}). \tag{3.2.2}$$

In particular the curvature of a Chern connection has type $(1, 1)$.

Formula (3.2.2) is particularly simple in the case of a line bundle, since then a local frame is just a local non-vanishing section. If s is such a holomorphic section and $h = \langle s, s\rangle$, then we obtain:

$$\vartheta = \partial \log h, \qquad \Theta = \bar\partial\partial \log h. \tag{3.2.3}$$

Also, since Θ gets conjugated under a change of frame, this has no effect in the case $k = 1$, since $GL(1, \mathbb{C})$ is abelian. Therefore Θ is a well-defined (purely imaginary) global 2-form on M.

Example 3.2.7. Let us compute the curvature of the Chern connection on the tautological line bundle J on $\mathbb{CP}^n$ (Example 3.2.2). Recall that J is a subbundle of the trivial line bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ and a hermitian metric is induced from the standard hermitian inner product on $\mathbb{C}^{n+1}$. We compute the curvature of the associated Chern connection in the chart $U_0 = \{z_0 \neq 0\}$ in which J is trivialised by

$$\varphi_0 : \mathbb{C}^n \times \mathbb{C} \to J$$
$$\varphi_0((w_1, \dots, w_n), u) = u(1, w_1, \dots, w_n) \in J|_{[1,w_1,\dots,w_n]}.$$

A non-vanishing holomorphic section is given by $s([1, z_1, \dots, z_n]) = (1, z_1, \dots, z_n)$ and so

$$h = \langle s, s\rangle = 1 + \sum_{i=1}^{n} |z_i|^2, \quad \text{and } \Theta = \bar\partial\partial \log h.$$

Hermitian metrics on complex manifolds

A particular holomorphic vector bundle associated to a complex manifold is $T^{1,0}M$, i.e. the holomorphic tangent bundle. A holomorphic frame on $T^{1,0}M$ can be given as $e_i = \frac{\partial}{\partial z_i}$ for local complex coordinates $z_1, \dots, z_n$ and a hermitian metric on $T^{1,0}M$ can be locally written as

$$h = \sum_{i,j} h_{ij} dz_i \otimes d\bar z_j$$

where $[h_{ij}]$ is a hermitian matrix. We can also view h as a $\mathbb{C}$-valued metric on $TM_{\mathbb{R}}$. Observe that

$$\operatorname{Re} h = \frac{1}{2} \sum_{i,j} h_{ij} dz_i d\bar z_j, \quad \text{and}$$

$$\operatorname{Im} h = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij} dz_i \wedge d\bar z_j.$$

Thus a hermitian metric on $T^{1,0}M$ gives us:

1) a Riemannian metric g on $TM_{\mathbb{R}}$ which satisfies $g(JX, JY) = g(X, Y)$, and

2) a non-degenerate 2-form $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$ called the *fundamental form* of g with $\omega(X,Y) = g(JX,Y)$.

Example 3.2.8. Suppose that $\dim_{\mathbb{C}} M = 1$, and $z = x+iy$ is a local coordinate. A hermitian metric on $T^{1,0}M$ is then written locally as $h\, dz \otimes d\bar{z}$ for a local function $h > 0$. The connection matrix of the Chern connection is $(\partial h)h^{-1} = \frac{\partial \log h}{\partial z} dz$, and the curvature matrix is

$$\Theta = \bar{\partial}\partial \log h = \frac{\partial^2 \log h}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = \left(-\frac{1}{4}\Delta \log h\right) dz \wedge d\bar{z},$$

where $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the usual Laplacian. The fundamental form of (M,h) is $\frac{\sqrt{-1}}{2} h\ dz \wedge d\bar{z}$ and hence

$$\Theta = -\sqrt{-1}\ \underbrace{\left(-\frac{\Delta \log h}{2h}\right)}_{\substack{\text{Gaussian curvature}\\ \text{of a surface}}}\ \omega.$$

Curvature of subbundles and quotient bundles

Definition 3.2.9. Let E be a hermitian holomorphic vector bundle on a complex manifold M. We say that a section $s \in \Gamma(\Lambda^{1,1}M \otimes \mathrm{Hom}(E,E))$ is positive[1] at $x \in M$ (notation: $s(x) > 0$) if $s(x)(v,\bar{v}) \in \mathrm{Hom}(E_x,E_x)$ is a positive definite hermitian matrix for every $v \in T^{1,0}M$. Similarly $s(x) \geq 0, s(x) < 0, s(x) \leq 0, s(x) \geq s'(x)$ etc. We write $s > 0$ etc. if $s(x) > 0$ etc. at every point $x \in M$.

Example 3.2.10. In example 3.2.7 we computed the curvature of the tautological bundle on $\mathbb{CP}^n$ with the metric induced from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ as given (on $z_0 \neq 0$) by

$$\Theta = \bar{\partial}\partial \log \left(1 + \sum_{i=1}^{n} |z_i|^2\right).$$

Hence:

$$\Theta(x)\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_i}\right) = \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \log\left(1 + \sum_{i=1}^{n} |z_i|^2\right) = \frac{\partial}{\partial z_i}\left(\frac{z_i}{1 + \sum\limits_{i=1}^{n} |z_i|^2}\right)$$

$$= \frac{1}{1 + \sum\limits_{i=1}^{n} |z_i|^2} - \frac{z_i \bar{z}_i}{\left(1 + \sum\limits_{i=1}^{n} |z_i|^2\right)^2} = \frac{1 + \sum\limits_{j \neq i} |z_j|^2}{\left(1 + \sum\limits_{i=1}^{n} |z_i|^2\right)^2} > 0.$$

Therefore $\Theta(x)\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) < 0$ for $i = 1, \ldots, n$ and all x, i.e. $\Theta < 0$.

---

[1] more precisely: Griffiths-positive

Let now F be a holomorphic subbundle of E and equip F with the induced hermitian metric. Write R_E and R_F for the curvatures of the respective Chern connections. These are sections of $\Lambda^{1,1}M \otimes \mathrm{Hom}(E,E)$ and $\Lambda^{1,1}M \otimes \mathrm{Hom}(F,F)$, respectively. Let $N = F^{\perp}$. This is a smooth complex subbundle of E. If $s \in \Gamma(F)$ and $t \in \Gamma(N)$, then

$$0 = d\langle s,t\rangle = \langle Ds,t\rangle + \langle s,Dt\rangle.$$

This means that in a frame of E consisting of a frame for F together with a frame for N, the connection matrix of $D = D_E$ has the form

$$\vartheta_E = \begin{pmatrix} \vartheta_F & A \\ -A^* & \vartheta_N \end{pmatrix},$$

where A is a matrix of $(1,0)$-forms. We compute the curvature matrix with respect to this decomposition:

$$\Theta_E = d\vartheta_E - \vartheta_E \wedge \vartheta_E = \left(\begin{array}{c|c} d\vartheta_F - \vartheta_F \wedge \vartheta_F + A \wedge A^* & \text{something} \\ \hline \text{something} & \text{something} \end{array}\right),$$

and conclude that the curvature matrix of D_F satisfies

$$\Theta_E|_F = \Theta_F + A \wedge A^*.$$

Since A has type $(1,0)$, $A \wedge A^* \geq 0$, and so $R_F \leq R_E|_F$, which means that the *curvature decreases in holomorphic subbundles.* In particular, if $E \simeq M \times \mathbb{C}^k$ is a trivial bundle equipped with the standard hermitian metric, so that $R_E \equiv 0$, then $R_F \leq 0$ for any holomorphic subbundles of E (as it was for the tautological bundle J). If we apply this to a submanifold M of $\mathbb{C}^n$ and $F = T^{1,0}M \subset T^{1,0}\mathbb{C}^n|_M$ with the induced hermitian metric, we conclude that the curvature of such $T^{1,0}M$ is always nonpositive. In particular if M is a Riemann surface locally embedded in $\mathbb{C}^n$ (as a complex submanifold), then its Gaussian curvature is nonpositive.

Observe that the same calculation for the quotient bundle $Q = E/F$ shows that $R_Q \geq R_E|_F$, i.e. the curvature increases in holomorphic quotient bundles. As an application suppose that a holomorphic vector bundle $E \xrightarrow{\pi} M$ is *generated by its sections*, i.e. there exist holomorphic sections $s_1, \ldots, s_l \in H^0(M,E)$, $l \geq \operatorname{rank} E$, such that $s_1(x), \ldots, s_l(x)$ generate E_x for every $x \in M$. This gives us a surjective (holomorphic) vector bundle homomorphism

$$M \times \mathbb{C}^l \to E, \qquad (x,u) \longmapsto \sum_{i=1}^{l} u_i s_i(x),$$

which can be interpreted as saying that E is a quotient bundle of a trivial bundle. If we equip E with the hermitian metric induced from the Euclidean metric on $M \times \mathbb{C}^l$, then we conclude that $R_E \geq 0$.

3.3 Chern classes of complex vector bundles

Let $E \overset{\pi}{\to} M$ be a complex vector bundle on a smooth manifold, and D an arbitrary connection on E. Its curvature R^D is a section of $\Lambda^2 M \otimes \mathrm{Hom}(E,E)$, which we can view as a matrix of 2-forms and speak of the trace $\operatorname{tr} R^D \in \Omega^2(M)$ of R^D. Recall the formula for the curvature matrix of D in a local frame: $\Theta = d\vartheta - \vartheta \wedge \vartheta$. Therefore $\operatorname{tr}\Theta = \operatorname{tr} d\vartheta = d \operatorname{tr}\vartheta$, and hence $\operatorname{tr} R^D$ is a closed 2-form[2], called the *Ricci form* of D.

Lemma 3.3.1. *The cohomology class* $[\operatorname{tr} R^D] \in H^2_{dR}(M) \otimes \mathbb{C}$ *does not depend on* D.

Proof. Let D and D' be two connections on E and set $A = D - D'$. Observe that it is a well-defined global section of $\Gamma(\Lambda^1 M \otimes \mathrm{Hom}(E,E))$, and therefore

$$\operatorname{tr} R^D - \operatorname{tr} R^{D'} = \operatorname{tr}(\Theta - \Theta') = d \operatorname{tr}(\vartheta - \vartheta') = d \operatorname{tr} A$$

is a globally defined exact 1-form. □

Remark 3.2.4 implies that $[\operatorname{tr} R^D]$ is purely imaginary.

Definition 3.3.2. The cohomology class $\frac{\sqrt{-1}}{2\pi}[\operatorname{tr} R^D] \in H^2_{\mathrm{dR}}(M)$ is called the *first Chern class* of E and is denoted by $c_1(E)$.

This is a topological invariant of a vector bundle.

Example 3.3.3. We compute $c_1(J_{\mathbb{CP}^1})$. Recall that we computed the curvature matrix of the Chern connection for the hermitian metric induced from $\mathbb{CP}^1 \times \mathbb{C}^2$ in the chart U_0 as

$$\Theta = \bar\partial\partial \log\bigl(1 + |z|^2\bigr) = \frac{1}{(1+|z|^2)^2} d\bar z \wedge dz.$$

Now, $H^2_{dR}(\mathbb{CP}^1)$ is identified with $\mathbb{C}$ via integration: $\omega \longmapsto \int\limits_{\mathbb{CP}^1} \omega$. We compute

$$\begin{aligned} c_1(J_{\mathbb{CP}^1}) &= \frac{\sqrt{-1}}{2\pi} \int\limits_{\mathbb{C}} \frac{1}{(1+|z|^2)^2}\, d\bar z \wedge dz \underset{z=re^{i\theta}}{=} \frac{1}{\pi} \int\limits_{[0,2\pi]\times[0,\infty)} \frac{r}{(1+r^2)^2}\, d\theta \wedge dr \\ &= -\frac{1}{\pi} \int\limits_0^\infty \int\limits_0^{2\pi} \frac{r}{(1+r^2)^2}\, d\theta\; dr = -1. \end{aligned}$$

Remark 3.3.4. This actually implies that $c_1(J_{\mathbb{CP}^n}) = -1$ for any n. Indeed, restricting $J_{\mathbb{CP}^n}$ to a $\mathbb{CP}^1 \subset \mathbb{CP}^n$ is just $J_{\mathbb{CP}^1}$, and hence $c_1(J_{\mathbb{CP}^n}) \cdot \mathbb{CP}^1 = -1$ for any $\mathbb{CP}^1$ in $\mathbb{CP}^n$. Since $H_2(\mathbb{CP}^n)$ is generated by such a $\mathbb{CP}^1$, the claim follows.

Proposition 3.3.5. *Let* E *and* F *be complex bundles of ranks* k *and* l *on a smooth manifold* M. *Then*

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[2] We cannot conclude that it is exact, since it is d(*something*) only locally.

(i) $c_1(\Lambda^k E) = c_1(E)$

(ii) $c_1(E \oplus F) = c_1(E) + c_1(F)$

(iii) $c_1(E \otimes F) = lc_1(E) + kc_1(F)$

(iv) $c_1(E^*) = -c_1(E)$

(v) $c_1(f^*E) = f^*c_1(E)$.

Proof. Use the induced connections on these bundles as defined in the homework. □

We now define higher Chern classes. The 1st one was defined using the trace of the curvature $R^D \in \Gamma(\Lambda^2 M \otimes \mathrm{Hom}(E, E))$. We view R^D as a matrix with entries in $\Lambda^2 M$, and observe that the even exterior algebra $\bigoplus \Lambda^{2i} M$ is commutative (with respect to the exterior product). Therefore we can consider polynomials in the entries of such a matrix, e.g. the determinant or the adjoint matrix. In particular, invariant polynomials make sense. Recall that if A is a $k \times k$ matrix, then we can set

$$\det(t + A) = \sum_{i=0}^{k} P_i(A) t^{k-i}$$

so that $P_1(A) = \operatorname{tr} A$, $P_k(A) = \det(A)$. Each P_i is a homogeneous polynomial of degree i in entries of A, invariant under conjugation. If we apply this to R^D we obtain closed forms

$$P_i(R^D) = P_i(\Theta) \in \Omega^{2i}(M).$$

Definition 3.3.6. The *i-th Chern class* of a complex vector bundle E over a smooth manifold M is the cohomology class

$$c_i(E) = \left[P_i \left(\frac{\sqrt{-1}}{2\pi} R^D \right) \right] \in H^{2i}_{dR}(M), \qquad i = 1, \dots, \operatorname{rank} E.$$

Once again, this does not depend on the choice of a connection D (exercise), and it is real, owing to Remark 3.2.4.

First Chern class of a line bundle

Let $L \xrightarrow{\pi} M$ be a complex line bundle on a smooth manifold M. Recall, from §2.1, that L is given by transition functions $g_{ij} : U_i \cap U_j \to GL(1, \mathbb{C})$, where $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of M. These transition functions satisfy

$$g_{ij} g_{ji} = 1, \quad g_{ij} g_{jk} g_{ki} = 1, \ \forall i, j, k \in I.$$

Since $GL(1, \mathbb{C}) \simeq \mathbb{C}^*$, the collection $\{g_{ij}\}$ can be viewed as a Čech cochain in $C^1\big(\mathcal{U}, (C^\infty)^*\big)$, where $(C^\infty)^*$ is the sheaf of nonvanishing complex-valued smooth functions on M. The above conditions on the g_{ij} imply that this cochain is a cocycle, i.e. $\delta(\{g_{ij}\}) = 0$. Moreover, if $\mathcal{U}'$ is another covering on which L is trivialised, then L is also trivialised on a common refinement of $\mathcal{U}$ and $\mathcal{U}'$. If $\{g_{ij}\}$ and $\{g'_{ij}\}$ are two sets of transition functions on this common refinement, then they correspond to the same line bundle if and only if there exist smooth nonvanishing functions f_i such that $g'_{ij} f_j = f_i g_{ij}$ for all i, j. This means that $g'_{ij} g_{ij}^{-1}$ is a Čech coboundary. We conclude:

Proposition 3.3.7. *Complex line bundles on M are in $1-1$ correspondence with elements of $\check{H}^1\big(M, (C^\infty)^*\big)$.* □

Remark 3.3.8. Line bundles form a group with respect to the tensor product. It is easy to see that the above correspondence is a group isomorphism.

We have an exact sequence of sheaves (cf. Example 2.3.9):

$$0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty \longrightarrow (C^\infty)^* \longrightarrow 0, \tag{3.3.1}$$

where the second map is $f \mapsto \exp(2\pi i f)$. The long exact sequence on cohomology, together with Proposition 2.4.8, yields an isomorphism

$$0 \to \check{H}^1\big(M, (C^\infty)^*\big) \longrightarrow \check{H}^2(M, \mathbb{Z}) \to 0. \tag{3.3.2}$$

Hence:

Proposition 3.3.9. *Complex line bundles on M are in $1-1$ correspondence with $\check{H}^2(M, \mathbb{Z})$.* □

The image of a line bundle L under the isomorphim (3.3.2) is called the *Euler class* of L, denoted by $e(L)$. The group $\check{H}^2(M, \mathbb{Z})$ is a topological invariant, isomorphic to the (second) *singular cohomology* of M. There is a natural map $\check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathbb{R}) \simeq H^2_{\mathrm{dR}}(M)$, given by tensoring with $\mathbb{R}$. It is an isomorphism on the free part of the $\mathbb{Z}$-module $\check{H}^2(M, \mathbb{Z})$ and it sends torsion elements (i.e. any $\mathbb{Z}/p\mathbb{Z}$-part) to 0. We can now identify the differential-geometric definition of the 1st Chern class with the purely topological notion of the Euler class:

Theorem 3.3.10. *Let $L \xrightarrow{\pi} M$ be a complex line bundle on a smooth manifold M. Then $c_1(L)$ is equal to the image of $e(L)$ in $H^2_{\mathrm{dR}}(M)$.*

Proof. We first work out the explicit form of isomorphism (3.3.2). Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover such that each $L|_{U_i}$ is trivial, and let g_{ij} be the corresponding transition functions. The 1-cocycle $\{g_{ij}\}$ determines the element of $H^1\big(M, (C^\infty)^*\big)$ correponding to L. In order to compute the connecting homomorphism, we follow its construction in the proof of Theorem 2.4.6. We may assume that $\mathcal{U}$ is fine enough so that each $U_i \cap U_j$ is contractible, and set $h_{ij} = (2\pi\sqrt{-1})^{-1} \log g_{ij}$ (defined uniquely up an additive integer). Then $\{g_{ij}\}$ is the image of $\{h_{ij}\} \in C^1(\mathcal{U}, C^\infty)$ under the exponential map. The Čech coboundary operator δ sends $\{h_{ij}\} \in C^1(\mathcal{U}, C^\infty)$ to $\{z_{ijk}\} \in C^2(\mathcal{U}, C^\infty)$, where

$$z_{ijk} = h_{ij} - h_{ik} + h_{jk} = \frac{1}{2\pi\sqrt{-1}} \big(\log g_{ij} + \log g_{jk} + \log g_{ki}\big).$$

This is the image of a cocycle in $C^2(\mathcal{U}, \mathbb{Z})$ representing $e(L) \in \check{H}^2(M, \mathbb{Z})$.

We now compare this with $c_1(L)$. Choose a connection D on L, compatible with some hermitian metric on L. Owing to Remark 3.2.4 we can assume that the connection "matrix" $\vartheta_i \in \Omega^1(U_i)$ for D on $L|_{U_i}$ is purely imaginary. The curvature R^D is now a global 2-form, given as $d\vartheta_i$ in each U_i. Recall (Remark 2.4.11) the de Rham isomorphism $\check{H}^q(M, \mathbb{R}) = H^q_{dR}(M)$, valid for every q. The proof of this works analogously to the proof of the Dolbeault theorem: we have exact sequences of sheaves:

$$0 \to \mathbb{R} \xrightarrow{d} C^\infty \xrightarrow{d} Z^1_d \to 0, \qquad 0 \to Z^1_d \xrightarrow{d} \Omega^1 \xrightarrow{d} Z^2_d \to 0,$$

which give us isomorphisms

$$\check{H}^2(M, \mathbb{R}) \simeq \check{H}^1(M, Z^1_d) \simeq \check{H}^0(M, Z^2_d)/d\check{H}^0(M, C^\infty) \simeq H^2_{\mathrm{dR}}(M).$$

Starting on the right, with $c_1(L)$, we get $\frac{\sqrt{-1}}{2\pi} R^D \in H^0(M, Z^2_d)$ which, from the construction of the connecting homomorphism, corresponds to the cocycle $\frac{\sqrt{-1}}{2\pi}\{\vartheta_i - \vartheta_j\} \in \check{H}^1(M, Z^1_d)$. The transformation law for the connection matrix implies that $\vartheta_i - \vartheta_j = d \log g_{ji} = -d \log g_{ij}$, and applying the connecting homomorphism once again, gives the cocycle is $-\frac{\sqrt{-1}}{2\pi}\{\log g_{ij} + \log g_{jk} + \log g_{ki}\}$ in $\check{H}^2(M, \mathbb{R})$. □

Remark 3.3.11. For a line bundle L, $-2\pi i c_1(L)$ is the cohomology class of the curvature R^D for any connection D. If $c_1(L) = 0$, then R^D is exact, i.e. $R^D = d\phi$ for a global 1-form ϕ. This means that the connection $D' = D - \phi$ has zero curvature, i.e. L admits a *flat* connection. Such a bundle is called flat, and the discussion in the paragraph after Proposition 3.3.9 shows that flat line bundles on M are classified[3] by torsion elements of $\check{H}^2(M, \mathbb{Z})$, i.e. those which become zero in $\check{H}^2(M, \mathbb{R})$. If M is compact[4], then the *universal coefficient theorem* implies that the torsion part of $\check{H}^2(M, \mathbb{Z})$ is isomorphic to the torsion part of $H_1(M, \mathbb{Z}) \simeq \pi_1(M)/[\pi_1(M), \pi_1(M)]$.

Further reading:

(i) Complex line bundles are classified by $\check{H}^2(M, \mathbb{Z})$. One can ask whether there exist geometric objects associated to $\check{H}^3(M, \mathbb{Z})$ (and higher)? The answer is yes; they are called (abelian) *gerbes*; see, e.g., M. Murray, *An Introduction to Bundle Gerbes*, in: "The many facets of geometry" (OUP 2010), also arXiv:0712.1651, or Y. Loizides, *Introduction to Gerbes*, at `http://personal.psu.edu/yxl649/Introduction%20to%20bundle%20gerbes.pdf`.

(ii) We have seen that line bundles correspond to elements of $\check{H}^1\big(M, (C^\infty)^*\big)$. The same argument, involving trivialisations and

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[3] As complex line bundles, not as gauge equivalence classes of (L, D).

[4] More generally, if M is of finite type, i.e. all homology groups $H_i(M, \mathbb{Z})$ are finitely generated.

cocycles, shows that vector bundles of rank k correspond to $\check{H}^1\big(M, \mathcal{G}_k\big)$, where $\mathcal{G}_k$ is the sheaf of *nonabelian* groups of $GL(k, \mathbb{C})$-valued functions. Nonabelian cohomology quickly becomes very abstract if one wants to go beyond H^1. After the previous comment, you can guess that $\check{H}^2\big(M, \mathcal{G}_k\big)$ is related to *nonabelian* gerbes. See p. 16 and following in: Ieke Moerdijk, *Introduction to the language of gerbes and stacks*, arXiv:math/0212266.

(iii) Flat vector bundles are a large research area, mainly because they are closely related to representations of the fundamental group of a manifold. See, e.g., O. Guichard, *An Introduction to the Differential Geometry of Flat Bundles and of Higgs Bundles*, in: "The Geometry, Topology and Physics of Moduli Spaces of Higgs Bundles" (World Scientific 2018), also at `http://irma.math.unistra.fr/~guichard/assets/files/intro-bdle-ims.pdf`.

3.4 Chern classes of holomorphic vector bundles

First Chern class of a holomorphic line bundle

Let M be a complex manifold, $\mathcal{O}$ the sheaf of holomorphic functions, and $\mathcal{O}^*$ the sheaf of non-vanishing holomorphic functions on M. The same argument which led to Proposition 3.3.9 proves:

Proposition 3.4.1. *Holomorphic line bundles on M are in $1-1$ correspondence with elements of $\check{H}^1(M, \mathcal{O}^*)$.* □

Just as for complex line bundles, holomorphic line bundles form a group with respect to the tensor product. The above bijection is a group isomorphism.

Definition 3.4.2. The group of (isomorphism classes of) holomorphic line bundles on a complex manifold M is called the *Picard group* of M, denoted by $\mathrm{Pic}(M)$.

We consider now the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0,$$

and the associated boundary map on cohomology

$$\check{H}^1(M, \mathcal{O}^*) \xrightarrow{\delta} \check{H}^2(M, \mathbb{Z}).$$

This is similar to (3.3.2), but this time δ does not have to be either injective or surjective. Observe, from the long exact cohomology sequence, that δ is injective iff $\check{H}^1(M, \mathcal{O}) = H^{0,1}_{\bar\partial}(M) = 0$, and it is surjective iff $\check{H}^2(M, \mathcal{O}) = H^{0,2}_{\bar\partial}(M) = 0$. Also, if L is a holomorphic line bundle, then $\delta(L)$ is still the Euler class of L as a complex line bundle. This follows from the fact that δ and (3.3.2) commute with the embedding $\check{H}^1(M, \mathcal{O}^*) \hookrightarrow \check{H}^1\big(M, (C^\infty)^*\big)$.

As an application, we can finally classify holomorphic line bundles on $\mathbb{P}^1$:

Proposition 3.4.3. *A holomorphic line bundle L on $\mathbb{P}^1$ is isomorphic to H^k, where H is the hyperplane bundle and $k = c_1(L) \in \mathbb{Z}$.*

Proof. In Ex. 1 on Homework 3 you have shown that $H^{0,1}_{\bar\partial}(\mathbb{P}^1) = H^{0,2}_{\bar\partial}(\mathbb{P}^1) = 0$. Therefore δ is an isomorphism. □

Remark 3.4.4. The line bundle H^k is usually denoted by $\mathcal{O}(k)$.

Remark 3.4.5. Since the map δ is group homomorphism, the set of (isomorphism classes of) holomorphic line bundles with $\delta(L) = 0$ is a subgroup of $\operatorname{Pic}(M)$, denoted by $\operatorname{Pic}^0(M)$. These are holomorphic line bundles such that the underlying *complex* line bundle is trivial. In Example 2.2.6 we have identified holomorphic structures on the trivial line bundle over an elliptic curve C. We can now restate the result of that example as: $\operatorname{Pic}^0(C) \simeq C$.

Prescribing the Ricci curvature of a Chern connection

Let $E \xrightarrow{\pi} M$ be a holomorphic vector bundle over a complex manifold. Recall that the curvature of the Chern connection for any hermitian metric has type $(1,1)$ and that the curvature matrix in the unitary frame is skew-hermitian. Therefore $\frac{\sqrt{-1}}{2\pi} R^D$ is hermitian, and hence $P_i\left(\frac{\sqrt{-1}}{2\pi} R^D\right)$ is a real (i,i)-form. Theorem 3.3.10 implies now that

$$c_i(E) \in H^{i,i}(M) \cap H^{2i}(M, \mathbb{Z}).$$

Here $H^{2i}(M,\mathbb{Z})$ really means the image of $H^{2i}(M,\mathbb{Z})$ in $H^2_{\mathrm{dR}}(M)$, i.e. $H^{2i}(M,\mathbb{Z})$ modulo torsion.

Let now φ be a closed $(1,1)$-form[5] with $[\varphi] = c_1(E)$. We ask: *does there exist a hermitian metric on E, such that the Ricci curvature (i.e.* $\operatorname{tr} R^D$*) of the associated Chern connection is $-2\pi i\varphi$?*

Let $\langle\, ,\,\rangle$ be an arbitrary hermitian metric on E. In a local holomorphic frame $(e_1, \ldots, e_k)$ with the associated matrix $h_{ij} = \langle e_i, e_j\rangle$ the curvature matrix of the Chern connection is given by the following formula (cf. (3.2.2)):

$$\Theta = \bar\partial(\partial h h^{-1}),$$

which means that the Ricci form $\operatorname{tr} R^D$ is represented in this local frame by

$$\bar\partial\partial \log \det h.$$

We now modify the metric $\langle\, ,\,\rangle$ by multiplying it by $e^{f/k}$, where f is a smooth real function on M and $k = \operatorname{rank} E$. The new matrix h' is given by

$$h'_{ij} = e^{f/k}\langle e_i, e_j\rangle,$$

and hence $\det h' = e^f \det h$. Therefore the Ricci forms of the two Chern connections are related by

$$\operatorname{tr} R^{D'} - \operatorname{tr} R^D = \bar\partial\partial f.$$

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[5] Observe that a closed $(1,1)$-form is also $\bar\partial$-closed.

Therefore we find a hermitian metric with $\operatorname{tr} R^{D'} = -2\pi i\varphi$, provided we can solve the equation

$$\bar{\partial}\partial f = -2\pi i\varphi - \operatorname{tr} R^D.$$

The right-hand side of this equation is a closed imaginary $(1,1)$-form cohomologous to 0:

$$[-2\pi i\varphi - \operatorname{tr} R^D] = -2\pi i[\varphi] + 2\pi i c_1(E) = 0.$$

Therefore the answer to our question is: we can prescribe the Ricci curvature of a Chern connection on complex manifolds which satisfy the following condition:

Any exact real $(1,1)$*-form* β *on* M *is of the form* $\sqrt{-1}\partial\bar{\partial}f$ *for a smooth function* $f : M \to \mathbb{R}$.

This condition is called the *global* $\partial\bar{\partial}$*-lemma* and a simple sufficient criterion is given by:

Lemma 3.4.6. *Let* M *be a complex manifold with* $H^{0,1}_{\bar{\partial}}(M) = 0$. *Then the global* $\partial\bar{\partial}$*-lemma holds on* M.

Proof. Since β is exact, there exists a real 1-form α such that $d\alpha = \beta$. We decompose α as $\tau + \tau'$, where τ has type $(1,0)$ and τ' $(0,1)$. It follows that $\tau' = \overline{\tau}$. We have

$$\beta = (\partial + \bar{\partial})(\tau + \overline{\tau}) = \underbrace{\partial\tau}_{(2,0)} + \underbrace{(\bar{\partial}\tau + \partial\overline{\tau})}_{(1,1)} + \underbrace{\bar{\partial}\overline{\tau}}_{(0,2)},$$

and therefore $\beta = \bar{\partial}\tau + \partial\overline{\tau}$, $\partial\tau = 0 = \bar{\partial}\overline{\tau}$. Since $H^{0,1}(M) = 0$, there exists a function $u : M \longrightarrow \mathbb{C}$ such that $\overline{\tau} = \bar{\partial}u$. Then $\tau = \partial\overline{u}$, and:

$$\beta = \bar{\partial}\tau + \partial\overline{\tau} = \bar{\partial}\partial\overline{u} + \partial\bar{\partial}u = \partial\bar{\partial}(u - \overline{u}) = 2i\partial\bar{\partial}(\operatorname{Im} u).$$

The claim follows with $f = 2\operatorname{Im} u$. □

Chern classes of a complex manifold

Definition 3.4.7. Let M be a complex manifold. The i-th Chern class $c_i(M)$ of M is $c_i(TM)$, where TM is the holomorphic tangent bundle, $i = 1, \ldots, \dim_{\mathbb{C}} M$.

Remark 3.4.8. It follows from Proposition 3.3.5 that $c_1(M) = c_1(K_M^*)$, i.e. the first Chern class of a complex manifold equals the first Chern class of its anti-canonical bundle.

Example 3.4.9. We can compute the first Chern class of a projective space:

$$c_1(\mathbb{CP}^n) = c_1(K^*_{\mathbb{CP}^n}) = c_1\left((J^*)^{\otimes n+1}\right) = (n+1)c_1(J^*) = n+1,$$

where we used the result of Example 3.3.3. For $n = 1$, we obtain $c_1(\mathbb{CP}^1) = 2$. This is just the Gauss-Bonnet theorem: for any oriented compact surface S and any hermitian metric on TS we have (cf. Ex. 3.2.8)

$$c_1(S) = \int_S \frac{i}{2\pi} R \underset{R=-iK}{=} \frac{1}{2\pi} \int_S K = \chi(S).$$

We wish to relate the first Chern class of a submanifold Y of M to $c_1(M)$. We have an exact sequence of holomorphic vector bundles on Y:

$$0 \longrightarrow TY \longrightarrow TM \longrightarrow TM/TY \longrightarrow 0. \tag{3.4.1}$$

The bundle TM/TY is called the *normal bundle* of Y in M, and is denoted by $N_{Y/M}$ or simply N_Y. If $\dim M = n$ and $\dim V = m$, then taking the highest exterior power shows that:

$$K_M^*|_Y \simeq K_Y^* \otimes \Lambda^{n-m} N_Y. \tag{3.4.2}$$

Therefore $c_1(Y) = c_1(M) - c_1(N_Y)$.

Of particular interest are complex manifolds with $c_1(M) = 0$. This condition is satisfied if the canonical bundle is trivial, i.e. there exists a non vanishing holomorphic n-form on M ($n = \dim_{\mathbb{C}} M$). Here are some examples:

Examples 3.4.10. 1. $\mathbb{C}^n$, but also quotients of $\mathbb{C}^n$ by discrete subgroups preserving the complex volume form $dz_1 \wedge \cdots \wedge dz_n$, e.g. quotients by lattices (complex tori).

2. The quadric $Q = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^2 + z_3^2 = 1\}$. Observe that this is a complexification of S^2, so that $H^2(Q) \neq 0$. The following holomorphic 2-form does not vanish on Q, and therefore trivialises K_Q:

$$z_1 dz_2 \wedge dz_3 + z_2 dz_3 \wedge dz_1 + z_3 dz_1 \wedge dz_2.$$

3. *Fermat hypersurface* of degree $n+1$ in a projective space:

$$V = \{[z_0, z_1, \dots, z_n] \in \mathbb{CP}^n \mid z_0^{n+1} + \cdots + z_n^{n+1} = 0\}.$$

Since V is defined by a homogeneous equation of degree $n+1$, i.e. by a section of H^{n+1}, the normal bundle N_V is isomorphic to $H^{n+1}|_V$ (see Homework 7). Since $K_{\mathbb{CP}^n} \simeq (H^{n+1})^*$, formula (3.4.2) shows that K_V is trivial.
For $n = 3$, this Fermat hypersurface is an example of the famous K3 surfaces (simply connected 2-dimensional complex manifolds with $c_1 = 0$).

We finish the section with a generalisation of the Gauss-Bonnet theorem:

Theorem 3.4.11 (Gauss-Bonnet-Chern theorem). *If M is a compact complex manifold with $\dim_{\mathbb{C}} M = n$, then $c_n(M) = \chi(M)$, i.e.*

$$\int_M \det\left(\frac{\sqrt{-1}}{2\pi} R^D\right) = \chi(M),$$

for any connection D on TM.

Sketch of a proof. Fix a hermitian metric $\langle\, ,\rangle$ on TM. We can find a (smooth) vector field X with finitely many zeros $p_1, \dots, p_k$. Let U_i be disjoint neighbourhoods of p_i such that TU_i is trivial. Find a function ϕ which is $\equiv 1$ on each $B_i(\epsilon) = \{m \in U_i; |X(m)| \leq \epsilon\}$ and $\equiv 0$ on $M \setminus \bigcup B_i(2\epsilon)$. Moreover ϵ should be small enough enough so that each $B_i(2\epsilon)$ is relatively compact in U_i.

Using partitions of unity construct a $\langle\, ,\rangle$-compatible connection ∇ with following properties:

(i) on $M\backslash\bigcup B_i(2\epsilon)$, ∇ preserves the orthogonal splitting $TM = \langle X\rangle \oplus E$ and the curvature of the line bundle $\langle X\rangle$ is identically zero;

(ii) for each $i = 1, \dots, k$, ∇ is flat on $B_i(\epsilon)$;

(iii) On each $B_i(2\epsilon)$ the connection matrix of ∇ is of the form $(1-\phi)\pi^*\Omega$, where $\pi : B_i(2\epsilon)\backslash\{p_i\} \to S^{2n-1}$ is the radial projection, and Ω is the connection matrix of the standard round metric on S^{2n-1}.

Condition (i) implies that $\det\left(\frac{\sqrt{-1}}{2\pi}R^\nabla\right)$ is identically 0 on $M\backslash\bigcup U_i$. On the other hand, conditions (ii) and (iii) imply that, for each $i = 1, \dots, k$, $\int_{U_i} \det\left(\frac{\sqrt{-1}}{2\pi}R^\nabla\right)$ is equal to the index of the vector field X at p_i. The result follows from the Poincaré-Hopf theorem. □

Remark 3.4.12. As the above proof suggests, the result is valid for any almost complex manifold. In fact it is true for any even-dimensional oriented manifold, provided we replace $c_n(M)$ with the Euler class of the tangent bundle.

Further reading:

(i) For a detailed proof of the Gauss-Bonnet-Chern theorem see the beautiful original paper of Chern (which started the whole characteristic classes theory): *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. (2) 45 (1944), 747–752; or the survey article *The Gauss-Bonnet-Chern Theorem on Riemannian Manifolds* by Yin Li, arXiv:1111.4972.

(ii) For more fun with Chern classes see §§3.3–3.4 in Griffiths & Harris.

3.5 Line bundles and divisors

In complex analysis an important role is played by meromorphic functions. We now define them on any complex manifold.

Definition 3.5.1. Let M be a complex manifold. A *meromorphic function* f on M is given locally as a quotient of two holomorphic functions, i.e. for some open covering $\{U_i\}_{i\in I}$ of M we have $f|_{U_i} = g_i/h_i$, where g_i and h_i are relatively prime[6] holomorphic functions on U_i, and $g_i h_j = g_j h_i$ on any $U_i \cap U_j$.

Remark 3.5.2. f is not really a function: it is not defined at points where $g_i = h_i = 0$. Strictly speaking, f is an equivalence class of $\{U_i, g_i, h_i\}$, where the equivalence relation is essentially given in the above definition. I shall leave the details to the more formally inclined among you.

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[6] As elements of the ring $\mathcal{O}(U_i)$ (which is a *GCD domain*), i.e. any holomorphic function which divides both g_i and h_i does not vanish on U_i.

We can now define meromorphic functions on any open subset of M, and, consequently, we have the (additive) sheaf $\mathcal{M}$ of meromorphic functions on M, as well as the (multiplicative) sheaf $\mathcal{M}^*$ of meromorphic functions which are not identically zero.

We now consider zeros and poles of a meromorphic function. Observe that the zero set of a holomorphic function f is not necessarily a submanifold (unless 0 is a regular value of f). In fact, we do not want to consider zeros of f as just a subset: as for holomorphic functions of one variable, we want to keep track of the multiplicities of zeros.

Definition 3.5.3. A subset V of M is called an *analytic hypersurface* if every point $p \in V$ has a neighbourhood U such that $V \cap U$ is the zero set of a holomorphic function $f \in \mathcal{O}(U)$ which divides every other function $g \in O(U)$ with $g|_{V \cap U} = 0$. f is called a *local defining function* near p. An analytic hypersurface is called *irreducible* if V cannot be written as a union of analytic hypersurfaces (i.e. the local defining functions cannot be factorised into holomorphic functions which have zeros on V).

A *divisor* D on M is a locally finite[7] formal linear combination

$$D = \sum k_i V_i$$

of irreducible analytic hypersurfaces with integer coefficients.

Clearly divisors form an abelian group with respect to addition, denoted by $\mathrm{Div}(M)$.

Let h be a holomorphic function on M and V an irreducible analytic hypersurface of M. Let $p \in V$ and let f be local defining function of V in a neighbourhood of p. We define the *order of h along V at p* to be the largest integer $k = k_{V,p}$ such that f^k divides h in a neighbourhood of p. Observe that $k_{V,p}$ is locally constant, and since an irreducible analytic hypersurface must be connected, $k_{V,p}$ is actually independent of p. We can therefore speak of the order of h along V, denoted $\mathrm{ord}_V(h)$. It is basically the order to which h vanishes along V. We now define the divisor (h) of h as $\sum \mathrm{ord}_V(h) V$, where the sum runs over all irreducible analytic hypersurfaces in M. This is a locally finite sum and (h) is well defined. Observe that if $\dim_{\mathbb{C}} M = 1$, then $(h) = \sum m_i z_i$, where z_i are distinct zeros of h and m_i is the multiplicity of z_i.

Similarly, if f is a meromorphic function with a local representation g/h, then we define the order of f along V to be $\mathrm{ord}_V(f) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h)$. The divisor (f) of f is then $\sum \mathrm{ord}_V(f) V$.

We have the following sheaf-theoretic interpretation of divisors:

Proposition 3.5.4. $\mathrm{Div}(M) \simeq \check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*)$, *i.e. divisors correspond to global sections of the sheaf* $\mathcal{M}^*/\mathcal{O}^*$.

Proof. A global section of $\mathcal{M}^*/\mathcal{O}^*$ is given by a (locally finite) open cover $\{U_i\}$ and meromorphic functions $f_i \in \mathcal{M}^*(U_i)$ such that on any $U_i \cap U_j$ $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$. This last condition means that on $U_i \cap U_j$ $\mathrm{ord}_V(f_i) = \mathrm{ord}_V(f_j)$,

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[7] I.e. any point has a neighbourhood which intersects only finitely many V_i.

for any V. Therefore the divisor $D = \sum \mathrm{ord}_V(f_i)V$ is well defined. Conversely, let $D = \sum k_i V_i$ be a divisor, and let $\{U_\alpha\}$ be an open cover such that only finitely many V_i intersect each U_α and each of these V_i has a local defining function $f_i \in \mathcal{O}(U_\alpha)$. Set $f_\alpha = \prod f_i^{k_i}$. This is a meromorphic function on U_α. Since the local defining functions are determined up to a nonvanishing factor, f_α is defined up to multiplication by an element of $\mathcal{O}^*(U_\alpha)$. Therefore (U_α, f_α) defines a global section of $\mathcal{M}^*/\mathcal{O}^*$. ☐

Remark 3.5.5. In algebraic geometry, elements of $\mathrm{Div}(M)$ are called *Weil divisors* and elements of $\check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*)$ are *Cartier divisors.* They do not coincide for more general (singular) spaces.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{O}^* \longrightarrow 0.$$

The long exact cohomology sequence reads:

$$0 \to \check{H}^0(M, \mathcal{O}^*) \to \check{H}^0(M, \mathcal{M}^*) \to \check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*) \to \check{H}^1(M, \mathcal{O}^*) \to \check{H}^1(M, \mathcal{M}^*) \to \dots$$

Since $\check{H}^1(M, \mathcal{O}^*)$ is the group of holomorphic line bundles on M, this means that there is a natural map associating a line bundle to a divisor. We can see this explicitly as follows. If D is a divisor with local defining functions[8] $f_i \in \mathcal{M}^*(U_i)$ for some open cover $\{U_i\}$, then $g_{ij} = f_i/f_j$ are holomorphic and nonvanishing on each $U_i \cap U_j$. Moreover $g_{ij}g_{jk}g_{ki} = 1$ on every triple intersection, and, hence g_{ij} are transition functions of a line bundle. It is easy to see that a different choice of $\{U_i\}$ and f_i gives an isomorphic line bundle. Moreover, this line bundle, denoted by $[D]$, is trivial if and only if there is a cover $\{U_i\}$ and functions $h_i \in \mathcal{O}^*(U_i)$ such that $g_{ij} = h_i/h_j$. But this means that $f_i h_i^{-1} = f_j h_j^{-1}$ on every $U_i \cap U_j$, so that f defined as $f_i h_i^{-1}$ on U_i is a global meromorphic function with $(f) = D$. Therefore $[D]$ is trivial if and only if D is the divisor of a meromorphic function.

Furthermore observe, directly from the definition of $[D]$, that the local data (U_i, f_i) defines a *meromorphic section* of $[D]$, simply because the functions f_i satisfy $f_i = g_{ij} f_j$ on each $U_i \cap U_j$. Conversely, if L is a holomorphic line bundle and s is a meromorphic section of L, i.e. there exist local meromorphic functions s_i which satisfy $s_i = g_{ij} s_j$ on the intersections, then the s_i define a divisor D with $L = [D]$. The divisor associated to a meromorphic section is denoted by (s).

Remark 3.5.6. This shows, in particular, that a line bundle is associated to a divisor if and only if it admits a meromorphic section. We shall prove later that this is the case for every line bundle on a projective manifold. In other words, if M is projective, then the map $\check{H}^0(M, \mathcal{M}^*/\mathcal{O}^*) \to \check{H}^1(M, \mathcal{O}^*)$ (i.e. $\mathrm{Div}(M) \to \mathrm{Pic}(M)$) is surjective. This is false in general; in fact, “most” compact complex manifolds do not admit any divisors, but many have nontrivial holomorphic line bundles.

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[8] If $D = \sum k_s V_s$ and h_s is a local defining function of V_s, then the local defining function of D is $\prod h_s^{k_s}$.

An even stronger property is the vanishing of $\check{H}^1(M, \mathcal{M}^*)$. This is not true even for projective manifolds: see the article "*The sheaf of nonvanishing meromorphic functions in the projective algebraic case is not acyclic*" by X. Chen, M. Kerr, and J.D. Lewis, *C. R. Acad. Sci. Paris, Ser. I*, 348 (2010), 291–293.

It perhaps also worth pointing out that $\mathcal{M}$ and $\mathcal{M}^*$ are not coherent sheaves (cf. Remark 2.3.12). In particular GAGA does not apply, so that the sheaf of algebraic meromorphic functions (local quotients of two polynomials) on projective manifolds is much smaller than $\mathcal{M}$.

We now wish to express $c_1([D])$ in terms of D. Observe that if V is an analytic hypersurface in M, then its set of singular points has (complex) codimension at least 2 in M. Therefore integration over V is well defined for forms with compact support, and we obtain a linear functional $\phi \to \int_V \phi$ on $H_c^{n-2}(M)$. Via Poincaré duality this corresponds to a cohomology class $[\eta_V] \in H^2_{\mathrm{dR}}(M)$. This *Poincaré dual* is characterised by

$$\int_M \eta_V \wedge \phi = \int_V \phi \quad \text{for any closed } (n-2)\text{-form } \phi \text{ with compact support.}$$

We can also integrate over formal linear combinations of V: just integrate over each V and take the corresponding linear combination of results. Therefore we can associate the Poincaré dual $[\eta_D]$ to any divisor D on M. We have:

Theorem 3.5.7. *Suppose that a holomorphic line bundle L on a complex manifold is of the form $L = [D]$ for some divisor D. Then $c_1(L) = [\eta_D] \in H^2_{\mathrm{dR}}(M)$.*

Proof. Let $D = \sum k_i V_i$ for a locally finite collection $\{V_i\}$ of irreducible analytic hypersurfaces. We need to show that the curvature form R of a Chern connection on $[D]$ satisfies

$$\frac{\sqrt{-1}}{2\pi} \int_M R \wedge \phi = \sum_i k_i \int_{V_i} \phi$$

for any compactly supported $(n-2)$-form ϕ. Since ϕ has compact support, and $\{V_i\}$ is locally finite, the right hand side is a finite sum for any such ϕ. Since c_1 is additive with respect to tensor product of line bundles, it is additive with respect to addition of divisors. It is therefore enough to prove this identity for $D = V$ - a single irreducible analytic hypersurface. Let us choose a hermitian metric on L. Then, for any nonvanishing local holomorphic section e of L, the curvature matrix Θ of the corresponding Chern connection is given by (cf. (3.2.3)):

$$\Theta = \bar{\partial}\partial \log |e|^2.$$

We can rewrite $\bar{\partial}\partial$ as dd', where $d' = \frac{1}{2}(\partial - \bar{\partial})$. Let $\{U_i, f_i\}$ be local defining data for V and let s be the corresponding holomorphic section of $[V]$ with $s^{-1}(0) = V$, i.e. $s = f_i$ on U_i. We consider a tubular neighbourhood of V given by

$$D(\epsilon) = \{m \in M;\ |s(m)| < \epsilon\},$$

and integrate, using the Stokes theorem:

$$\int_M R \wedge \phi = \lim_{\epsilon \to 0} \int_{M \setminus D(\epsilon)} dd' \log |s|^2 \wedge \phi = \lim_{\epsilon \to 0} \int_{\partial D(\epsilon)} d' \log |s|^2 \wedge \phi.$$

On each U_i we can write $|s|^2 = h_i f_i \bar{f}_i$, for some positive real function h_i. We can replace each U_i with an open subset, which is relatively compact in U_i, and therefore we can assume that $d'h_i$ is bounded on $D(\epsilon) \cap U_i$. Consequently:

$$\lim_{\epsilon \to 0} \int_{\partial D(\epsilon) \cap U_i} d' \log h_i \wedge \phi = 0.$$

It follows that

$$\lim_{\epsilon \to 0} \int_{\partial D(\epsilon) \cap U_i} d' \log |s|^2 \wedge \phi = \lim_{\epsilon \to 0} \sqrt{-1}\, \mathrm{Im} \int_{\partial D(\epsilon) \cap U_i} \partial \log f_i \wedge \phi.$$

In a neighbourhood of a smooth point p of $V \cap U_i$ we can find holomorphic coordinates $(w_1, \ldots, w_n)$ with $w_1 = f_i$, and $w' = (w_2, \ldots, w_n)$ holomorphic coordinates on V. We can write the form ϕ as

$$\phi = g(w)\omega + \psi, \text{ where } \omega = dw_2 \wedge \cdots \wedge dw_n \wedge d\bar{w}_2 \wedge \cdots \wedge d\bar{w}_n,$$

and every term in ψ contains either dw_1 or $d\bar{w}_1$. Then

$$\mathrm{Im}\, \partial \log f_i \wedge \phi = \mathrm{Im} \frac{dw_1}{w_1} \wedge (g(w)\omega + \psi) = \mathrm{Im} \frac{dw_1}{w_1} \wedge g(w)\omega.$$

Then, in a neighbourhood U of p

$$\mathrm{Im} \lim_{\epsilon \to 0} \int_{\partial D(\epsilon) \cap U} \partial \log f_i \wedge \phi = \mathrm{Im} \lim_{\epsilon \to 0} \int_{\partial D(\epsilon) \cap U} \frac{dw_1}{w_1} \wedge g(w)\omega =$$
$$= \mathrm{Im} \lim_{\epsilon \to 0} \int_{|w_1| = \epsilon/\sqrt{h_i}} \frac{dw_1}{w_1} \wedge g(w)\omega = \mathrm{Im} \lim_{\epsilon \to 0} \Big(\int_{|w_1| = \epsilon/\sqrt{h_i}} \frac{dw_1}{w_1} \wedge g(0, w')\omega + O(\epsilon) \Big) =$$
$$= -\,\mathrm{Im} \lim_{\epsilon \to 0} \int_{w'} \Big(\int_{|w_1| = \epsilon/\sqrt{h_i}} \frac{dw_1}{w_1} \Big) g(0, w')\omega = -2\pi \int_{w'} g(0, w')\omega = -2\pi \int_{V \cap U} \phi.$$

Therefore

$$\frac{\sqrt{-1}}{2\pi} \int_M R \wedge \phi = \int_V \phi,$$

which concludes the proof. □

Chapter 4

Kähler manifolds

4.1 Kähler metrics

Recall from §3.2 that a hermitian manifold is a complex manifold M with a hermitian metric on $T^{1,0}M$ or equivalently a Riemannian metric g on TM (now denoting the real tangent bundle) which is invariant with respect to the complex J, i.e.:

$$g(JX, JY) = g(X, Y) \quad \forall X, Y \in \Gamma(TM).$$

The bundles TM and $T^{1,0}M$ are isomorphic, via $X \mapsto X - iJX$, and we have two connections on TM associated to g: the Chern connection D and the Levi-Civita connection ∇. Both of them are compatible with g:

$$\begin{cases} d(g(X,Y)) = g(DX, Y) + g(X, DY) \\ d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y) \end{cases} \qquad \forall X, Y \in \Gamma(TM).$$

In addition, D satisfies $D^{0,1} = \bar{\partial}$, which can be rephrased as

$$D_Z(JX) = JD_Z X \quad \forall X, Z \in \Gamma(TM),$$

or simply as $DJ = 0$. On the other hand, the Levi-Civita connection ∇ is torsion free:

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y \in \Gamma(TM).$$

Clearly, hermitian metrics such that these two connections coincide should be interesting. First of all, let us give several other equivalent conditions:

Theorem 4.1.1. *Let g be a hermitian metric on a complex manifold (M, J). Then the following are equivalent*

i) J is parallel for the Levi-Civita connection;

ii) D has zero torsion;

iii) the Levi-Civita and the Chern connections coincide;

iv) The fundamental form ω of g is closed (recall that $\omega(X,Y) = g(JX,Y)$);

v) For all $p \in M$ there exists a smooth real function f in a neighbourhood U of p such that $\omega|_U = i\partial\bar{\partial}f$;

vi) Around each point $p \in M$ there exist holomorphic coordinates $w = (w_1, \dots, w_n)$, such that

$$g_w\Big(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}\Big) = \delta_{ij} + O(|w|^2).$$

Proof. Note that conditions i), ii), and iii) are equivalent owing to the uniqueness of Chern and Levi-Civita connections. We are going to show that i) $\implies$ iv) $\implies$ v) $\implies$ vi) $\implies$ i).

i) $\implies$ iv): Since $\nabla g = 0$ and $\nabla J = 0$, $\nabla \omega = 0$. Every parallel form is closed, however, due to the identity:

$$d\alpha(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i (\nabla_{X_i}\alpha)(X_0, \dots, \hat{X}_i, \dots, X_p), \quad \forall \alpha \in \Omega^p(M).$$

iv) $\implies$ v): Let U be a neighbourhood of p biholomorphic to a polydisk $\mathbb{C}^n$. Since $\omega|_U$ is exact, the claim follows from Lemma 3.4.6 (the $\partial\bar{\partial}$-lemma).

v) $\implies$ vi): In local complex coordinates around $p \in M$ we can write:

$$\omega = i \sum_{l,m} \omega_{lm} dz_l \wedge d\bar{z}_m,$$

where

$$\omega_{lm} = \frac{1}{2}\delta_{lm} + \sum_j (a_{jlm} z_j + b_{jlm}\bar{z}_j) + O(|z|^2).$$

Since ω is real, $a_{jlm} = \overline{b}_{jlm}$. It follows from v) that

$$a_{jlm} = \frac{\partial^3 f}{\partial z_j \partial z_l \partial \bar{z}_m},$$

which implies that $a_{jlm} = a_{ljm}$ for all j, l, m. Set $w_m = z_m + \sum\limits_{l,m} a_{jlm} z_j z_l$ and compute:

$$\begin{aligned} \frac{i}{2}\sum_m dw_m \wedge d\bar{w}_m = \frac{i}{2}\sum_m dz_m \wedge d\bar{z}_m + i \sum_{l,m,j} a_{jlm} z_j \; dz_l \wedge d\bar{z}_m + \\ + i \sum_{l,m,j} \overline{a}_{jlm}\bar{z}_j \; dz_m \wedge d\bar{z}_l + O(|z|^2) = \\ = i \sum_{l,m} \omega_{lm} \; dz_l \wedge d\bar{z}_m + O(|z|^2) = \omega + O(|w|^2), \end{aligned}$$

which is equivalent to vi).

vi) $\implies$ i): Let $p \in M$ and let $w_i = x_i + \sqrt{-1}\, y_i$ be the coordinates around p found in vi). Since the Christoffel symbols of the Levi-Civita connection ∇ depend only on the first derivatives of the metric tensor, they are equal to zero at p. Consequently $\nabla J|_p = 0$. Since p is arbitrary, $\nabla J = 0$ on M.

□

Definition 4.1.2. A hermitian metric on a complex manifold satisfying the equivalent conditions i) – vi) is called a *Kähler metric.*
Its fundamental form is called the *Kähler form*, and the local function in v) is the *Kähler potential.* Local coordinates having the property in vi) are called *normal Kähler coordinates.*

Remark 4.1.3. Yet another equivalent definition of a Kähler metric is that its holonomy is a subgroup of $U(n)$ (this is an equivalent formulation of i)).

Examples 4.1.4. 1) Standard metric on $\mathbb{C}^n$

$$g = \frac{1}{2}\operatorname{Re}\left(\sum_s dz_s \otimes d\bar{z}_s\right).$$

Its fundamental form is

$$\omega = \frac{i}{2}\sum_s dz_s \wedge d\bar{z}_s = \frac{i}{2}\partial\bar{\partial}|z|^2,$$

and, hence, $f(z) = \frac{1}{2}|z|^2$ is a global Kähler potential $f : \mathbb{C}^n \to \mathbb{R}$. Note that g and J are invariant under the standard action of $U(n)$ on $\mathbb{C}^n$.

2) The *Fubini-Study metric* on $\mathbb{CP}^n$:
For $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1}\backslash\{0\}$ set $\omega = i\partial\bar{\partial}\log\bigl(|z|^2\bigr)$ and observe that ω is invariant under rescalings $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$. Therefore ω defines a real, closed $(1,1)$-form on $\mathbb{CP}^n$.
Now set $g(X,Y) = \omega(X, JY)$, and observe that condition iv) of the above theorem implies that g is a Kähler metric provided it is positive definite. We compute g in the chart $U_0 = \{z_0 \neq 0\}$ with local coordinates $w_i = \frac{z_i}{z_0}$. Write

$$\omega = i\partial\bar{\partial}\log\bigl(1+|w|^2\bigr) = \partial\left(\frac{i}{1+|w|^2}\sum_{s=1}^{n} w_s d\bar{w}_s\right) =$$
$$= \frac{i}{1+|w|^2}\sum_{s=1}^{n} dw_s \wedge d\bar{w}_s - \frac{i}{(1+|w|^2)^2}\left(\sum_{s=1}^{n}\bar{w}_s dw_s\right) \wedge \left(\sum_{s=1}^{n} w_s d\bar{w}_s\right).$$

Since both ω and J are invariant under the action of $U(n+1)$, it is enough to check that g is positive definite at one point, say $p = [1, 0, \dots, 0]$, i.e. $w = 0$. But $\omega|_p = i\sum\limits_{s=1}^{n} dw_s \wedge d\bar{w}_s$, and so $g|_p = \sum\limits_{s=1}^{n} dw_s d\bar{w}_s$, which shows that g is positive definite. This Kähler metric is called the *Fubini-Study metric.*

Remark 4.1.5. Recall that $\mathbb{CP}^n = S^{2n+1}/S^1$. The Fubini-Study metric is the quotient metric of the round metric on S^{2n+1}.

Once we have these two basic examples, we obtain plenty more, since:

Proposition 4.1.6. *A complex submanifold of Kähler manifold, equipped with the induced metric, is Kähler.*

Proof. Let $(N, J, g_N) \subset (M, J, g_M)$ be as in the statement. The fundamental form ω_N of g_N is just the pullback (restriction) of the fundamental form ω_M of g_M, hence closed. □

Therefore every complex projective manifold, as well as a complex submanifold of $\mathbb{C}^n$ (i.e. a Stein manifold), has at least one Kähler metric. Observe also that the product of Kähler manifolds is again Kähler. On the other hand many complex manifolds do not admit any Kähler metric. An example of an obstruction is given by:

Proposition 4.1.7. *If M is a compact Kähler manifold, then*

$$H^{2q}_{dR}(M) \neq 0 \quad \textit{for all} \quad q \leq n = \dim_{\mathbb{C}} M.$$

Proof. Let ω be a Kähler form on M. Then ω^q is a closed form, which I claim cannot be exact. Indeed, had we $\omega^q = d\psi$, then $\omega^n = d\left(\psi \wedge \omega^{n-q}\right)$ and then

$$\mathrm{vol}(M) = \int_M \omega^n = \int_M d\left(\psi \wedge \omega^{n-q}\right) = 0,$$

which is impossible. □

Thus, for example, there are no Kähler metrics on Hopf manifolds which we defined in Chapter 1 (these are diffeomorphic to $S^1 \times S^{2n-1}$, $n \geq 2$). Another topological restriction is as follows:

Proposition 4.1.8. *Let M be a compact Kähler manifold. Then the identity map on q-forms induces an injective map*

$$H^{q,0}_{\bar\partial}(M) \to H^q_{dR}(M),$$

i.e. every nonzero holomorphic q-form is closed and never exact.

Proof. Let η be a holomorphic q-form. In a local unitary frame $\{\varphi_i\}$, we can write it as

$$\eta = \sum_{|I|=q} f_I \varphi_I, \quad \varphi_I = \varphi_{i_1 \ldots i_q} = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_q}.$$

Then

$$\eta \wedge \overline{\eta} = \sum_{I,J} f_I \overline{f}_J \varphi_I \wedge \overline{\varphi}_J.$$

On the other hand

$$\omega = \frac{\sqrt{-1}}{2}\sum_i \varphi_i \wedge \overline{\varphi}_i \implies \omega^{n-q} = c_q \sum_{|K|=n-q} \varphi_K \wedge \overline{\varphi}_K.$$

Hence

$$\eta \wedge \overline{\eta} \wedge \omega^{n-q} = c_q' \sum_{|I|=q} |f_I|^2 \varphi_I \wedge \overline{\varphi}_I \wedge \varphi_{I^c} \wedge \overline{\varphi}_{I^c},$$

where I^c denotes the complement of I, since the only nonzero wedge products arise when $K \cap I = \emptyset$ and $K \cap J = \emptyset$, which implies that $I = J$. Therefore

$$\eta \wedge \overline{\eta} \wedge \omega^{n-q} = c_q'' \left(\sum_{|I|=q} |f_I|^2 \right) \omega^n.$$

In particular, if $\eta \neq 0$, then the integral of $\eta \wedge \overline{\eta} \wedge \omega^{n-q}$ over M is nonzero. If, however, $\eta = d\psi$, then

$$\eta \wedge \overline{\eta} \wedge \omega^{n-q} = d\left(\psi \wedge \overline{\eta} \wedge \omega^{n-q}\right),$$

since $d\omega = 0$ and $d\overline{\eta} = d\left(d\overline{\psi}\right) = 0$, and we obtain a contradiction. Therefore a nonzero holomorphic form cannot be exact. To show that it is closed, observe that $d\eta = (\partial + \bar{\partial})\eta = \partial\eta$, which means that $d\eta$ is an exact holomorphic $(q+1)$-form, and the previous argument implies that $d\eta = 0$. $\square$

4.2 Hodge decomposition

The last result is a particular case of a much stronger fact, which is known as the *Hodge decomposition.*

Theorem 4.2.1. *On a compact Kähler manifold, the following relations hold:*

$$H^r_{dR}(M,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_{\bar{\partial}}(M), \qquad H^{p,q}_{\bar{\partial}}(M) = \overline{H^{q,p}_{\bar{\partial}}}(M).$$

Remark 4.2.2. Thus, on a compact Kähler manifold (in particular on any projective manifold), the Dolbeault cohomology can be viewed as a refinement of the the de Rham cohomology.

Remark 4.2.3. 1) In particular, all odd Betti numbers $b_{2s+1}(M) = \dim H^{2s+1}_{dR}(M)$ are even.

2) The theorem fails badly for noncompact Kähler manifolds. Recall that we showed (example 1.6.7) that $\dim H^{0,1}_{\bar{\partial}}\left(\mathbb{C}^2\backslash\{0\}\right) = \infty$. On the other hand: $H^1_{dR}\left(\mathbb{C}^2\backslash\{0\}\right) = 0$.

Outline of the proof. The proof is completely analogous to that of the Riemannian Hodge theorem. I shall outline it, since I am not sure that everyone took the course “Mannigfaltigkeiten”.

Let V be a vector space with an inner product. There is an induced inner product on each tensor power $V^{\otimes k}$, $k \geq 1$, and, by restriction, on each exterior power $\Lambda^k V$. If $(e_1, \dots, e_n)$ is an orthonormal basis on V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ is an orthonormal basis of $\Lambda^k V$. If (M, g) is a Riemannian manifold, then we can do this on each $T_x^* M$, so we get an inner product on each $\Lambda^k T_x^* M$. If (M, g) is *oriented*, i.e. we have a nonvanishing volume form dV, and *compact*, then we can define an inner product on differential forms in $\Omega^k(M)$:

$$\langle \alpha, \beta \rangle = \int\limits_M \langle \alpha|_x, \beta|_x \rangle dV.$$

We seek, in every cohomology class in $H^k_{dR}(M)$, a representative with the smallest norm. How to find such an element?

We can view each cohomology class $[\psi] \in H^k_{dR}(M)$ as an affine space $P = \{\psi + d\eta \mid \eta \in \Omega^{k-1}(M)\}$. If M is compact, then $\Omega^k(M)$ with the above inner product is a pre-Hilbert space[1], and were P a closed subspace, we could find an element of the smallest norm by the orthogonal projection, using the decomposition $\Omega^k(M) = d\Omega^{k-1}(M) \oplus \left(d\Omega^{k-1}(M)\right)^{\perp}$. The orthogonal projection can be expressed via the adjoint operator to d:

$$\|\psi + d\eta\|^2 = \|\psi\|^2 + \|d\eta\|^2 + 2\langle \psi, d\eta \rangle = \|\psi\|^2 + \|d\eta\|^2 + 2\langle d^*\psi, \eta \rangle.$$

Therefore, if $d^*\psi = 0$, then ψ has the smallest norm in P. Thus we conclude that cohomology classes should be represented by forms ψ such that $d\psi = 0$ and $d^*\psi = 0$. We need to understand the operator

$$d^* : \Omega^{k+1} \to \Omega^k.$$

Let us look again at the inner product on $\Lambda^k V$. If $(e_1, \dots, e_n)$ is an oriented orthonormal basis, then we can define a linear isomorphism

$$\begin{aligned} * :\ & \Lambda^k V \to \Lambda^{n-k} V \quad \text{via} \\ & \omega \wedge *\tau = \langle \omega, \tau \rangle\ e_1 \wedge \cdots \wedge e_n \quad \forall \omega,\ \tau \in \Lambda^k V. \end{aligned}$$

The operator $*$ is called the *Hodge dual.* In particular, $*1 = e_1 \wedge \cdots \wedge e_n$, $\langle *\tau_1, *\tau_2 \rangle = \langle \tau_1, \tau_2 \rangle$, $* \circ * = (-1)^{k(n-k)}$ on $\Lambda^k V$. Now observe that $*d*$ maps $\Omega^{k+1}(M)$ to $\Omega^k(M)$ and

$$\langle d\alpha, \beta \rangle dV = d\alpha \wedge *\beta = d(\alpha \wedge *\beta) - (-1)^k \alpha \wedge d * \beta,$$

---

[1] i.e. it does not have be complete, just as continuous functions on a interval with the L^2-norm.

so that

$$\langle d\alpha, \beta\rangle dV - d(\alpha \wedge *\beta) = (-1)^{k+1}\alpha \wedge *\beta = (-1)^{k+1}(-1)^{k(n-k)}\alpha \wedge *^2 d * \beta = \\ = -(-1)^{nk}\langle \alpha, *d*\beta\rangle dV.$$

After integration we obtain

$$\langle d\alpha, \beta\rangle = \langle \alpha, (-1)^{nk+1} * d * \beta\rangle,$$

which means that $d^* = (-1)^{nk+1} * d*$ is the adjoint operator of d, called the *codifferential*. A form ω such that $d^*\omega = 0$ is called *co-closed*.

Thus we need to show that, on a compact oriented Riemannian manifold (M, g), any cohomology class has a representative ψ with $d^*\psi = 0$ (and $d\psi = 0$). Observe that such a form automatically satisfies $(dd^* + d^*d)\psi = 0$. The operator

$$\Delta = \Delta_g = dd^* + d^*d : \ \Omega^k(M) \to \Omega^k(M)$$

is called the *Riemannian Laplacian*, or the *Laplace-Beltrami operator*. On functions in $\mathbb{R}^n$:

$$\begin{aligned} \Delta f &= (dd^* + d^*d)f = d^*df = - * d * df = - * d * \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\right) \\ &= - * d \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} * dx_j\right) \underset{d*dx_j=0}{=} - * \left(\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_i \wedge *dx_j\right) \\ &= - * \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} dV \underset{*dV=1}{=} - \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}. \end{aligned}$$

In general, if a Riemannian metric in local coordinates has a form $g = \sum_{i,j} g_{ij} dx_i dx_j$, then

$$\Delta_g f = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \left(\frac{\partial}{\partial x_i}\left(\sqrt{|g|}\ g^{ij} \frac{\partial f}{\partial x_j}\right)\right),$$

where $[g^{ij}] = [g_{ij}]^{-1}$ and $|g| = \det[g_{ij}]$. A form ψ such that $\Delta\psi = 0$ is called *harmonic*. Clearly $d\psi = 0$ and $d^*\psi = 0$ imply that ψ is harmonic. On a compact manifold we also have the converse:

Lemma 4.2.4. *If M is compact, then any harmonic form ψ satisfies $d\psi = d^*\psi = 0$.*

Proof.

$$0 = \int_M \langle \Delta\psi, \psi\rangle dV = \int_M \langle dd^*\psi + d^*d\psi, \psi\rangle dV = \int_M \left(|d\psi|^2 + |d^*\psi|^2\right) dV.$$

□

Corollary 4.2.5. *A harmonic function on an oriented compact connected Riemannian manifold is constant.*

Let $\mathcal{H}^k_\Delta(M)$ denote the vector space of harmonic k-forms, i.e:

$$\mathcal{H}^k_\Delta(M) = \{\psi \in \Omega^k(M) \mid \Delta\psi = 0\}.$$

Theorem 4.2.6 (Hodge–de Rham)**.** *On a compact oriented Riemannian manifold (M, g) we have*

$$\Omega^k(M) = \mathcal{H}^k_\Delta(M) \oplus d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M),$$

where the summands are mutually orthogonal with respect to $\langle\cdot,\cdot\rangle$.

Before discussing a proof, let us look at some applications:

Corollary 4.2.7. *The natural map $f : \mathcal{H}^k_\Delta(M) \to H^k_{dR}(M)$, given by $\psi \longmapsto [\psi]$, is an isomorphism.*

Proof. Since $d\psi = 0$, the map is well-defined. Since $\mathcal{H}^k_\Delta$ is orthogonal to exact forms, the kernel of f is trivial. Finally, let $[\omega] \in H^k_{dR}(M)$ and decompose $\omega = \omega^H + d\lambda + d^*\mu$, where ω^H is harmonic. Then

$$0 = \langle d\omega, \mu\rangle = \langle dd^*\mu, \mu\rangle = \langle d^*\mu, d^*\mu\rangle.$$

Hence $d^*\mu = 0$ and $[\omega] = [\omega^H + d\lambda] = [\omega^H]$, which means that f is surjective. □

Corollary 4.2.8 (Poincaré duality)**.** *On a compact oriented n-manifold M*

$$H^k_{dR}(M) \simeq H^{n-k}_{dR}(M).$$

Proof. Put any Riemannian metric on M. The corresponding Hodge dual operator $*$ gives an isomorphism $\mathcal{H}^k_\Delta(M) \simeq \mathcal{H}^{n-k}_\Delta(M)$. □

Remark 4.2.9. This isomorphism depends on the choice of a Riemannian metric on M. On a connected M, a better statement is that there exists a *natural* isomorphism $H^k_{dR}(M) \simeq H^{n-k}_{dR}(M)^*$, given by the pairing $(\phi, \psi) \mapsto \int_M \phi \wedge \psi$.

Idea of a proof of the Hodge-de Rham theorem: It is clear that the three summands $\mathcal{H}^k_\Delta(M)$, $d\Omega^{k-1}(M)$, $d^*\Omega^{k+1}(M)$ are mutually orthogonal: if ω is harmonic, then $\langle\omega, d\varphi\rangle = \langle d^*\omega, \varphi\rangle = 0$ and similarly $\langle\omega, d^*\mu\rangle = 0$. Moreover $\langle d\varphi, d^*\mu\rangle = \langle dd\varphi, \mu\rangle = 0$.

The hard part is to show that the direct sum is all of $\Omega^k(M)$. The solution is to complete $\Omega^k(M)$ with respect to a norm $\sum\limits_{i=0}^{s} \left|\nabla^i\psi\right|^2$, where ∇ is the covariant derivative, for some high order s. This is the *Sobolev space* $W^k_s(M)$, and it is a Hilbert space.

The Laplacian extends to a Fredholm operator[2]

$$\Delta_s : W_s^k(M) \to W_{s-2}^k(M) \text{ with } \operatorname{Ker}\Delta_s = \operatorname{Ker}\Delta,$$

which means that every "Sobolev class" harmonic form is smooth. We now have a well defined closed subspace $Y = (\operatorname{Ker}\Delta_s)^\perp \subset W_s^k(M)$ and we need to show that

$$Y \cap \Omega^k(M) = d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M).$$

We observe that

$$Y = \operatorname{Im}\Delta_s^* : W_{s-2}^k(M) \to W_s^k(M),$$

but on smooth forms $\Delta_s^* = \Delta$ (since Δ is self-adjoint), and therefore any smooth form orthogonal to $\operatorname{Ker}\Delta$ lies in the image of Δ, i.e.

$$\psi = \Delta u = (dd^* + d^*d)u = d(d^*u) + d^*(du) \in d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M). \quad \square$$

We can now obtain an analogous decomposition on a compact hermitian manifold (M, g, J) using the operator $\bar\partial$. We define the formal adjoint

$$\bar\partial^* : \Omega^{p,q+1}(M) \to \Omega^{p,q}(M)$$

and the $\bar\partial$-Laplacian $\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$. The key facts are:

- there is a natural orientation on a complex manifold;
- there is a hermitian inner product on each $\Omega^{p,q}(M)$;
- the Hodge star maps $\Omega^{p,q}(M)$ to $\Omega^{n-q,n-p}(M)$, where $n = \dim_{\mathbb{C}} M$;
- since $\dim_{\mathbb{R}} M$ is even, $*^2 = (-1)^{p+q}$;
- $\bar\partial^* = - * \partial *$.

A differential form φ satisfying $\Delta_{\bar\partial}\varphi = 0$ is called *$\bar\partial$-harmonic*. Again, on a compact M, $\Delta_{\bar\partial}\varphi = 0$ if and only if $\bar\partial\varphi = 0$ and $\bar\partial^*\varphi = 0$. We denote by $\mathcal{H}_\Delta^{p,q}(M)$ the space of $\bar\partial$-harmonic forms of type (p, q).

Theorem 4.2.10 (Hodge decomposition for the Dolbeault cohomology)**.** *On a compact hermitian manifold (M, g, J):*

$$\Omega^{p,q}(M) = \mathcal{H}_\Delta^{p,q}(M) \oplus \bar\partial\Omega^{p,q-1}(M) \oplus \bar\partial^*\Omega^{p,q+1}(M),$$

where the summands are orthogonal with respect to the global hermitian product $\langle\cdot,\cdot\rangle$.

Proof. The proof is completely analogous to that of the Hodge–de Rham theorem. $\square$

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[2] i.e. a bounded linear operator with finite-dimensional kernel and cokernel.

Corollary 4.2.11. *On a compact complex n-dimensional manifold M*

$$H^{p,q}_{\bar\partial}(M) \simeq H^{n-q,n-p}_{\bar\partial}(M).$$

Proof. The same as in Corollary 4.2.8. $\square$

Remark 4.2.12. In addition, complex conjugation induces an antilinear isomorphism $H^{p,q}_{\bar\partial}(M) \simeq H^{q,p}_{\bar\partial}(M)$.

On a hermitian manifold (M, g, J) we have defined two Laplacians: the Riemannian Δ_g and the complex $\Delta_{\bar\partial}$. In general, there is no relation between harmonic and $\bar\partial$-harmonic forms (otherwise we would have a relation between the de Rham and the Dolbeault cohomology). However:

Proposition 4.2.13. *If (M, g, J) is a Kähler manifold, then $\Delta_g = 2\Delta_{\bar\partial}$.*

Proof. Both Laplacians, when written in local coordinates, involve only first derivatives of the metric. Therefore in *normal Kähler coordinates* (complex coordinates in which the metric is Euclidean $+O(|z|^2)$) around a point p, the two Laplacians have the form

$$\Delta_{g-\text{euclidean}} + O(|z|^2) \text{ and } \Delta_{\bar\partial-\text{euclidean}} + O(|z|^2).$$

A simple calculation shows that $\Delta_{g-\text{euclidean}} = 2\Delta_{\bar\partial-\text{euclidean}}$, and so $\Delta_g|_p = 2\Delta_{\bar\partial}|_p$. Since p is arbitrary, the result follows. $\square$

Remark 4.2.14. This proof illustrates a general method of proving many results for Kähler manifolds. Any identity which holds on $\mathbb{C}^n$, and it involves only the metric and its first derivatives, is valid on any Kähler manifold.

On a compact Kähler manifold, we now obtain the Hodge relations from this proposition, the Hodge–de Rham theorem and theorem 4.2.10.

Further reading:

(i) As we have seen, not every complex manifold admits a Kähler metric. One can ask whether there are weaker conditions on a hermitian metric, which can be satisfied on any (compact) complex manifold. An example of such are the *Gauduchon metrics*, where the fundamental form ω satisfies $\partial\bar\partial\omega^{n-1} = 0$ ($n = \dim_{\mathbb{C}} M$). There exists a Gauduchon metric in every conformal class of a given hermitian metric. Examples of stronger (but weaker than Kähler) conditions are: $\partial\bar\partial\omega^{n-2} = 0$ (*astheno-Kähler*), $\partial\bar\partial\omega = 0$ (*strong Kähler with torsion* or *pluriclosed*); they can no longer be fulfilled on an arbitrary complex manifold.
A nice paper on astheno-Kähler manifolds is: A. Fino and A. Tomassini, "*On Astheno-Kähler metrics*", J. London Math. Soc. 83 (2011), 290–308, also at arXiv:0806.0735.
For a relation between Gauduchon metrics and Aeppli cohomology (see (iii) below): R. Piovani, A. Tomassini, "*Aeppli cohomology and Gauduchon metrics*", Complex Anal. Oper. Theory 14, 22 (2020). https://doi.org/10.1007/s11785-020-00984-6, also at arXiv:1909.02842.

(ii) For a more detailed proof of the Hodge theorem see, e.g. Griffiths & Harris.

(iii) The Hodge theorem can be interpreted as saying than on compact Kähler manifolds de Rham cohomology can be computed from the Dolbeault cohomology. There are several other cohomology theories on complex manifolds, which on non-Kähler manifolds are closer to the de Rham cohomology than the Dolbeault cohomology. Two (relatively) important ones are the *Aeppli cohomology* and the *Bott-Chern cohomology*. See the thesis of D. Angella *"Cohomological aspects of non-Kähler manifolds"*, arXiv:1302.0524, in particular Theorem 1.25.

4.3 Kodaira-Serre duality and Kodaira-Akizuki-Nakano vanishing theorem

Let E be a holomorphic vector bundle on a compact complex manifold M. Recall (§2.2, in particular Remark 2.2.7) that we have a well-defined operator $\bar\partial : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying $\bar\partial^2 = 0$, and, consequently, well-defined Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(M,E)$. If we now choose hermitian metrics on M and on E, then we obtain a hermitian metric on any $\Lambda^{p,q}(E) = \Lambda^{p,q}(M) \otimes E$. We can therefore define an inner product on $\Omega^{p,q}(E)$:

$$\langle \phi, \psi \rangle = \int_M \langle \phi|_m, \psi|_m \rangle dV.$$

We also have an operator

$$\wedge : \Lambda^{p,q}(E) \times \Lambda^{r,t}(E) \to \Lambda^{p+t,q+r}(M), \quad (\eta \otimes s) \wedge (\eta' \otimes s') = \langle s, s' \rangle \eta \wedge \overline{\eta'}.$$

We can now define the E-star operator $*_E : \Lambda^{p,q}(E) \to \Lambda^{n-q,n-p}(E)$ by the relation:

$$\langle \phi, \psi \rangle = \int_M \phi \wedge *_E \psi, \quad \forall \phi \in \Lambda^{p,q}(E).$$

Again we obtain an adjoint operator $\bar\partial^* = - *_E \partial *_E$ on E-valued differential forms, and can define the $\bar\partial$-Laplacian as before. We have the space $\mathcal{H}^{p,q}_{\Delta}(M,E)$ of harmonic (p,q)-forms, and the proof of the Hodge theorem goes through without any essential changes. Therefore

$$H^{p,q}_{\bar\partial}(M,E) \simeq \mathcal{H}^{p,q}_{\Delta}(M,E), \quad \forall\, p,q. \tag{4.3.1}$$

The corresponding "Poincaré duality" statement (cf. Corollaries 4.2.8 and 4.2.11) reads now:

$$H^{p,q}_{\bar\partial}(M,E) \simeq H^{n-q,n-p}_{\bar\partial}(M,E).$$

As in Remark 4.2.9, this isomorphism depends on the choice of metrics on M and on E. If M is connected, we can use instead a pairing between E-valued and E^*-valued differential forms, and obtain a canonical isomorphism:

$$H^{p,q}_{\bar\partial}(M,E) \simeq H^{n-p,n-q}_{\bar\partial}(M,E^*)^*.$$

Using the Dolbeault theorem for E-valued forms (Theorem 2.4.12), we can rephrase this as follows (from now on, I shall omit the "check" over cohomology groups of sheaves):

Theorem 4.3.1 (Kodaira-Serre duality)**.** *Let $E \xrightarrow{\pi} M$ be a holomorphic vector bundle on a connected compact complex manifold. There exist natural isomorphisms*

$$H^q\big(M, \mathcal{H}^{p,0}(E)\big) \simeq H^{n-q}\big(M, \mathcal{H}^{n-p,0}(E^*)\big)^*.$$

In particular, for $p = 0$:

$$H^q\big(M, \mathcal{O}(E)\big) \simeq H^{n-q}\big(M, \mathcal{O}(E^* \otimes K_M)\big)^*,$$

where $\mathcal{O}(E)$ denotes the sheaf of holomorphic sections of E. □

Example 4.3.2. Let C be a (connected) compact Riemann surface, i.e. a compact complex manifold of dimension 1. The Kodaira-Serre duality implies that $H^0(C, \mathcal{O}(K_C)) \simeq H^1(C, \mathcal{O})^*$, i.e. the dimension of the space of global holomorphic 1-forms equals the dimension of $H^1(C, \mathcal{O}) \simeq H^{0,1}_{\bar\partial}(C)$. Since C is Kähler (any hermitian metric is Kähler by dimensional reasons), the Hodge relations imply that $\dim H^{0,1}_{\bar\partial}(C) = \frac{1}{2}b_1(C)$. If you think a moment about a 2-dimensional compact real manifold with g holes, you can see that $g = \frac{1}{2}b_1(C)$. Therefore the dimension of the space of global holomorphic 1-forms is equal to the genus of C.

In general, $\dim H^0(C, \mathcal{O}(K_C))$ is called the *arithmetic genus* of C, and for more general (singular) algebraic curves it does not have to be equal to the topological genus (indeed, the latter may be not well defined).

Recall now (Definition 3.2.9) that we introduced the concept of positivity (or negativity) of curvature of a Chern connection on a hermitian holomorphic vector bundle. We consider the case of a line bundle, and call a holomorphic line bundle L *positive* if it admits a hermitian metric such that the curvature of the corresponding Chern connection is positive. We shall prove:

Theorem 4.3.3 (Kodaira-Akizuki-Nakano vanishing theorem)**.** *Let $L \to M$ be a positive line bundle on an n-dimensional compact complex manifold. Then*

$$H^q\big(M, \mathcal{H}^{p,0}(L)\big) = 0 \ \text{ if } p + q > n.$$

Remark 4.3.4. The Ricci form of a Chern connection is always closed. Therefore, if the Ricci form is positive, then it defines a Kähler metric. Consequently, a manifold which admits a positive line bundle is Kähler.[3]

Before proving the theorem we need some preparation. We define an operator[4] $L : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M)$ by $L(\eta) = \eta \wedge \omega$, and its adjoint

$$\Lambda = L^* = *^{-1} \circ L \circ * : \Omega^{p,q}(M) \to \Omega^{p-1,q-1}(M).$$

---

[3] Later we shall see that it is even projective.

[4] known as *Lefschetz operator*; Λ is known as the dual Lefschetz operator.

Lemma 4.3.5 (Kähler identities)**.** *Let M be a complex manifold equipped with a Kähler metric g. Then the following identities hold true:*

(i) $[\Lambda, L] = (n-p-q)\,\mathrm{Id}$*;*

(ii) $[\bar\partial, L] = [\partial, L] = 0$ *and* $[\bar\partial^*, \Lambda] = [\partial^*, \Lambda] = 0$*;*

(iii) $[\bar\partial^*, L] = i\partial$*,* $[\partial^*, L] = -i\bar\partial$ *and* $[\Lambda, \partial] = i\bar\partial^*$*,* $[\Lambda, \bar\partial] = -i\partial^*$*.*

Proof. We are going to prove (ii) and (iii). The proof of (i) requires a substantial detour into representation theory; see Griffiths and Harris, pp. 118–121, for details.
We compute for $\alpha \in \Omega^{p,q}(M)$:

$$[\bar\partial, L](\alpha) = \bar\partial(\omega\wedge\alpha) - \omega\wedge\bar\partial\alpha = (\bar\partial\omega)\wedge\alpha + \omega\wedge\bar\partial\alpha - \omega\wedge\bar\partial\alpha = 0,$$

since $\bar\partial\omega = 0$ (ω is closed), and similarly for $[\partial, L]$. Now

$$\begin{aligned}[\bar\partial^*, \Lambda](\alpha) = [-*\partial*, *^{-1}L*](\alpha) &= -*\partial L*\alpha + *^{-1}L*^2\,\partial*\alpha = \\ &= -*\partial L*\alpha + *L\partial*\alpha = -*[\partial, L]*\alpha = 0,\end{aligned}$$

and similarly for the last identity in (ii).

For (iii), notice first that the second identity is obtained by conjugation from the first one, and the remaining two are just the adjoints of the first two. Therefore we only need to prove the first identity $[\bar\partial^*, L] = i\partial$. Since this identity involves only the metric and its first derivatives, it is enough to prove it on $\mathbb{C}^n$ (cf. Remark 4.2.14). Moreover, since both sides are $\mathbb{C}$-linear, we only need to check the identity on monomials of the form $\alpha = f dz_I \wedge d\bar z_J$, where I, J are multi-indices.

Now, on $\mathbb{C}^n$, we have in standard coordinates $\omega = \frac{i}{2}\sum_{i=1}^n dz_i\wedge d\bar z_i$ and

$$\bar\partial^*\alpha = -2\sum_k \frac{\partial f}{\partial z_k} i_{\frac{\partial}{\partial\bar z_k}}(dz_I\wedge d\bar z_J),$$

where $i_{\frac{\partial}{\partial\bar z_k}}$ denotes interior multiplication (contraction)[5]. Then:

$$\begin{aligned}[\bar\partial^*, L]\alpha &= \bar\partial^*(\omega\wedge\alpha) - \omega\wedge\bar\partial^*\alpha \\ &= -2\sum_k \frac{\partial f}{\partial z_k} i_{\frac{\partial}{\partial\bar z_k}}(\omega\wedge dz_I\wedge d\bar z_J) + 2\sum_k \frac{\partial f}{\partial z_k}\omega\wedge i_{\frac{\partial}{\partial\bar z_k}}(dz_I\wedge d\bar z_J) \\ &= -2\sum_k \frac{\partial f}{\partial z_k}\Big((i_{\frac{\partial}{\partial\bar z_k}}\omega)\wedge dz_I\wedge d\bar z_J + \omega\wedge i_{\frac{\partial}{\partial\bar z_k}}(dz_I\wedge d\bar z_J) - \omega\wedge i_{\frac{\partial}{\partial\bar z_k}}(dz_I\wedge d\bar z_J)\Big) \\ &= -2\sum_k \frac{\partial f}{\partial z_k}(i_{\frac{\partial}{\partial\bar z_k}}\omega)\wedge dz_I\wedge d\bar z_J = -\sum_k i\frac{\partial f}{\partial z_k} dz_k\wedge dz_I\wedge d\bar z_J = i\partial\alpha,\end{aligned}$$

where we used $i_{\frac{\partial}{\partial\bar z_k}}\omega = -\frac{i}{2}dz_k$. □

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[5] Recall the identity $i_X(\alpha\wedge\beta) = (i_X\alpha)\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge(i_X\beta)$.

We now extend the operators L and Λ to act on forms with values in a holomorphic hermitian vector bundle:

$$L(\eta \otimes s) = (\omega \wedge \eta) \otimes s, \quad \eta \in \Omega^{p,q}(M), \ \ s \in H^0(E),$$

and similarly for Λ.

Lemma 4.3.6. *Let D be the Chern connection on a holomorphic hermitian vector bundle E over a Kähler manifold M. Then $[\Lambda, \bar\partial] = -i\big(D^{1,0}\big)^*$.*

Proof. Choose a local unitary frame on E, and let A be the connection matrix for D in this frame. Then

$$D^{1,0} = \partial + A^{1,0}, \quad D^{0,1} = \bar\partial + A^{0,1},$$

and the $\bar\partial$ in the desired formula is $D^{0,1}$ (i.e. the holomorphic structure on E). It follows that

$$\big(D^{1,0}\big)^* = \partial^* - (A^{1,0})^*.$$

Hence

$$[\Lambda, D^{0,1}] + i\big(D^{1,0}\big)^* = [\Lambda, \bar\partial] + [\Lambda, A^{0,1}] + i\partial^* - i(A^{1,0})^* = [\Lambda, A^{0,1}] - i(A^{1,0})^*,$$

where we used the Kähler identity from Lemma 4.3.5(iii). For any $p \in M$ we can find a local unitary frame such that the connection matrix A is zero at p. Therefore the left-hand side of the last formula vanishes identically on M. □

Proof of the Kodaira-Akizuki-Nagano theorem. Owing to (4.3.1) and to Dolbeault's theorem (Prop. 2.4.12) we have $H^q\big(M, \mathcal{H}^{p,0}(L)\big) = \mathcal{H}^{p,q}_\Delta(M, L)$. The assumption implies that there exists a hermitian metric on L such that $\omega = iR^D$ is the fundamental form of a Kähler metric on M. Therefore it is enough to show that there are no nonzero harmonic L-valued forms of degree $> n$. Let $\eta \in \mathcal{H}^{p,q}_\Delta(M, L)$. Then

$$R^D \wedge \eta = D^2\eta = (D^{1,0}\bar\partial + \bar\partial D^{1,0})\eta = \bar\partial D^{1,0}\eta,$$

since $\bar\partial\eta = 0$. Therefore

$$\begin{aligned} i\langle \Lambda R^D \wedge \eta, \eta\rangle = i\langle \Lambda\bar\partial D^{1,0}\eta, \eta\rangle \underset{\text{Lemma 4.3.6}}{=} i\big\langle \big(\bar\partial\Lambda - i(D^{1,0})^*\big) D^{1,0}\eta, \eta\big\rangle = \\ = i\langle \Lambda D^{1,0}\eta, \bar\partial^*\eta\rangle + \langle D^{1,0}\eta, D^{1,0}\eta\rangle \underset{\bar\partial^*\eta=0}{=} \langle D^{1,0}\eta, D^{1,0}\eta\rangle \geq 0. \end{aligned}$$

Similarly

$$\begin{aligned} i\langle R^D \wedge \Lambda\eta, \eta\rangle = i\langle (D^{1,0}\bar\partial + \bar\partial D^{1,0})\Lambda\eta, \eta\rangle = i\langle D^{1,0}\bar\partial\Lambda\eta, \eta\rangle + i\langle D^{1,0}\Lambda\eta, \bar\partial^*\eta\rangle = \\ = i\langle D^{1,0}\bar\partial\Lambda\eta, \eta\rangle = i\big\langle D^{1,0}\big(\Lambda\bar\partial + i(D^{1,0})^*\big)\eta, \eta\big\rangle = -\langle D^{1,0}(D^{1,0})^*\eta, \eta\rangle = \\ = -\langle (D^{1,0})^*\eta, (D^{1,0})^*\eta\rangle \leq 0. \end{aligned}$$

Since $iR^D = \omega$, the operator $iR^D \wedge (\,\cdot\,)$ is the Lefschetz operator L, and we obtain

$$0 \leq i\langle \Lambda R^D \wedge \eta, \eta\rangle - i\langle R^D \wedge \Lambda\eta, \eta\rangle = \langle [\Lambda, L]\eta, \eta\rangle \underset{\text{Lemma 4.3.5}}{=} (n-p-q)\|\eta\|^2.$$

Hence $p + q > n$ implies that $\eta = 0$. $\square$

Remark 4.3.7. Using the Kodaira-Serre duality, we conclude that if $L \xrightarrow{\pi} M$ is a negative line bundle, then $H^q\big(M, \mathcal{H}^{p,0}(L)\big) = 0$ if $p + q < n$.

Example 4.3.8. Observe that the Fubini-Study metric on $\mathbb{CP}^n$ is nothing else but iR^D, where D is a Chern connection on the hyperplane bundle $\mathcal{O}(1)$ (cf. Example 3.2.7). Therefore $\mathcal{O}(1)$ is a positive line, and so are its positive tensor powers $\mathcal{O}(m)$, $m \geq 1$. It follows from Theorem 4.3.3 that $H^q\big(\mathbb{CP}^n, \mathcal{H}^{p,0} \otimes \mathcal{O}(m)\big) = 0$ for $m > 0$ and $p + q > n$. In particular, since $\mathcal{H}^{n,0} = K_{\mathbb{CP}^n} = \mathcal{O}(-1-n)$ from Prop. 2.1.4, $H^q\big(\mathbb{CP}^n, \mathcal{O}(m)\big) = 0$ for $q > 0$ and $m \geq -n$. Using the Kodaira-Serre duality, we can deduce that $H^q\big(\mathbb{CP}^n, \mathcal{O}(m)\big) = 0$ if the integers n, q, m satisfy one of the following: (i) $0 < q < n$; (ii) $q = 0$, $m < 0$; (iii) $q = n$, $m > -n-1$.

The remaining cohomology groups are also easily computed: as in Ex. 1(b) in Homework 4, one shows that $H^0\big(\mathbb{CP}^n, \mathcal{O}(m)\big)$, $m > 0$, is isomorphic to the vector space of homogeneous polynomials of degree m in $n + 1$ variables. The Kodaira-Serre duality computes then $H^n\big(\mathbb{CP}^n, \mathcal{O}(m)\big)$ $(m > -n-1)$.

Remark 4.3.9. I have already mentioned that on $\mathbb{CP}^n$, $n > 1$, not every vector bundle splits into a direct sum of line bundles. First of all observe that any line bundle on $\mathbb{CP}^n$ is of the form $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$. The argument here is the same as for $\mathbb{CP}^1$ (Prop. 3.4.3) since $H^1_{\mathrm{dR}}(\mathbb{CP}^n) = 0$ and $H^2_{\mathrm{dR}}(\mathbb{CP}^n) \simeq \mathbb{C} \simeq H^{1,1}_{\bar\partial}(\mathbb{CP}^n)$. The last example gives now a necessary cohomological condition for a vector bundle $E \xrightarrow{\pi} \mathbb{CP}^n$ to split: $H^q\big(\mathbb{CP}^n, E \otimes \mathcal{O}(j)\big) = 0$ for $0 < q < n$ and all $j \in \mathbb{Z}$. It turns out that this condition is also sufficient. This is known as the *Horrocks criterion*; see the book by Okonek et al., cited at the end of §2.1.

Remark 4.3.10. In §3.4 we showed that if a complex manifold satisfies the global $\partial\bar\partial$-lemma, then the first Chern class of any vector bundle can be represented by the Ricci curvature of a Chern connection. Definition 3.2.9 can be used for arbitrary $(1,1)$-forms, and we say that $c_1(L) > 0$ if there is a form ϕ such that $[\phi] = c_1(L)$ and $-i\phi > 0$. Therefore, if the global $\partial\bar\partial$-lemma holds on M, then a line bundle L on M is positive if and only if $c_1(L) > 0$. In the homework you are asked to prove that the global $\partial\bar\partial$-lemma holds on any compact Kähler manifold. Therefore we can rephrase the assumption in the Kodaira-Akizuki-Nakano theorem as: M is Kähler and $c_1(L) > 0$.

Further reading:

(i) For other vanishing theorems see §VII.1-VII.9 of Demailly's book, cited at the end of Chapter 1.

(ii) Those of you who are more algebraic-minded, may want to ask (and some of you did) which results of the last two sections can be proved without recourse to analysis (for projective manifolds). Clearly not those where differential operators are essential in the statement, e.g. Theorem 4.2.6. But what about the Hodge decomposition theorem (Theorem 4.2.1)? Using Dolbeault's theorem, this can be rephrased without mentioning the operator $\bar{\partial}$. The answer is yes, at least for the first relation in the statement (i.e. the decomposition, not the one about conjugation). I believe it was Grothendieck who first suggested to prove it using l-adic cohomology, and this was done 20 years later by Deligne and Illusie (in 1987). The same methods lead to an algebraic proof of the Kodaira-Akizuki-Nakano vanishing theorem. A nice clear reference (but probably far beyond the scope of any course offered by the Institute for Algebraic Geometry; maybe a seminar?) is the book *Lectures on vanishing theorems* by H. Esnault and E. Viehweg (Birkhäuser 1992).

The Kodaira-Serre duality has many algebraic proofs, which can be found in most textbooks on algebraic geometry, e.g. in Hartshorne.

4.4 Holomorphic sectional curvature

The curvature of a connection D on a vector bundle E can be viewed as a 2-form with values in $Hom(E, E)$. In the special case $E = TM$, we can view the curvature as a $(3, 1)$-tensor (*Riemann curvature tensor*):

$$R^D : TM \times TM \times TM \to TM, \quad (X, Y, Z) \longmapsto R^D(X, Y)Z.$$

If D is the Levi-Civita connection of a Riemannian metric g, then one defines the *sectional curvature*, which associates a scalar to each tangent plane: if $X, Y \in T_pM$ are orthonormal, then the sectional curvature of the plane π spanned by X and Y is defined by

$$K(\pi) = K(X \wedge Y) = g(R^D(X, Y)Y, X).$$

$K(\pi)$ can be interpreted as the Gaussian curvature at p of the (immersed) 2-dimensional submanifold of M obtained by taking all geodesics with tangent directions belonging to π. In the course "Riemannian Geometry" we have seen that K determines R^D, and its study leads to many interesting topics and results: spaces of constant sectional curvature, pinching theorems, etc.

On a Kähler (or more generally, hermitian) manifold (M, g, J) there are special planes in tangent spaces: those invariant under J, i.e. having a basis X, JX. We define the *holomorphic sectional curvature* of (M, g, J) to be the sectional curvature restricted to the complex planes T_pM:

$$K(X \wedge JX) = g(R^D(X, JX)JX, X), \quad \text{with} \quad g(X, X) = 1.$$

An argument similar to that for the ordinary sectional curvature shows that on a Kähler manifold the holomorphic sectional curvature also determines the Riemannian curvature R^D.[6]

The only complete simply-connected n-dim Riemann manifolds with constant sectional curvature are S^n, $\mathbb{R}^n$, and the hyperbolic space H^n. We now ask for a similar classification of Kähler manifolds with constant holomorphic sectional curvature.

Theorem 4.4.1. *The Fubini-Study metric on $\mathbb{CP}^n$ has constant (positive) holomorphic sectional curvature.*

Proof. Recall that the fundamental form of the Fubini-Study metric is given by

$$\omega = i\partial\bar\partial \log |z|^2,$$

where $z \in \mathbb{C}^{n+1}\backslash\{0\}$ represents $[z] \in \mathbb{CP}^n$. Hence, as observed before, the Fubini-Study metric is invariant under the transitive action of $U(n+1)$ on $\mathbb{CP}^n$. I claim that $U(n+1)$ actually acts transitively on the space of all complex tangent planes ("holomorphic lines") in $T\mathbb{CP}^n$.

At a point $p \in \mathbb{CP}^n$, the space of all complex lines in $T_p\mathbb{CP}^n$ is $\mathbb{P}(T_p\mathbb{CP}^n) \simeq \mathbb{CP}^{n-1}$. The tangent space $T_p\mathbb{CP}^n$ is identified, from the definition of $\mathbb{CP}^n$, with $\mathbb{C}^{n+1}/L$, where $L = [p]$. Hence a *line* in $T_p\mathbb{CP}^n$ corresponds to a 2-dimensional subspace E of $\mathbb{C}^{n+1}$ such that $L \subset E$. Therefore the total space of all holomorphic lines in $T\mathbb{CP}^n$, i.e. $\mathbb{P}(T\mathbb{CP}^n)$, is the manifold $F_{1,2}$ of so-called $(1,2)$-*flags*:

$$L \subset E \subset \mathbb{C}^{n+1}, \text{ where } \dim L = 1, \ \dim E = 2.$$

It is a homogeneous manifold: the Gram-Schmidt orthogonalisation implies that

$$F_{1,2} = U(n+1)/\big(U(1) \times U(1) \times U(n-1)\big),$$

and hence $U(n+1)$ does act transitively on the space of all complex tangent planes. Therefore the holomorphic sectional curvature of the Fubini-Study metric is constant. In order to see that it is positive, consider a $\mathbb{CP}^1 \subset \mathbb{CP}^n$, say $\mathbb{P}(\langle e_0, e_1\rangle) \subset \mathbb{CP}^n$. It is the fixed point set of the subgroup

$$\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & U(n-1) \end{pmatrix}$$

of $U(n+1)$, and therefore totally geodesic. This means that the sectional curvature of $\pi \simeq T_p\mathbb{CP}^1 \subset T_p\mathbb{CP}^n$ is equal to the sectional curvature of π in $\mathbb{CP}^1$. But the latter is just the Gaussian curvature of the 2-sphere, hence positive. □

What about constant negative holomorphic sectional curvature? This would be an analogue of the hyperbolic space H^n and is even easier to construct: take the open unit ball

$$D_n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n; \, |z| < 1\}$$

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[6] For a proof see Prop. 7.1 in Ch. IX of Kobayashi and Nomizu.

and define a Kähler metric by the global Kähler potential $K(z) = -\log\left(1 - |z|^2\right)$, i.e. $\omega = i\partial\bar\partial K$. The metric is easily computed as

$$ds^2 = \frac{(1-|z|^2)\sum\limits_s dz_s d\overline{z}_s + \left(\sum\limits_s \overline{z}_s dz_s\right)\left(\sum\limits_s z_s d\overline{z}_s\right)}{\left(1-|z|^2\right)^2}.$$

Observe that this is clearly $U(n)$-invariant. It is actually $U(n,1)$-invariant, where the group $U(n,1)$ is defined as

$$U(n,1) = \left\{ A \in GL(n+1,\mathbb{C}) \;\middle|\; U^* \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -1 \end{pmatrix} U = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

i.e. the group of linear transformations preserving the indefinite hermitian distance $\sum\limits_{i=1}^{n} |z_i|^2 - |z_{n+1}|^2$. If we write

$$U = \begin{pmatrix} A & B \\ C & d \end{pmatrix}, \text{ where } A \in \mathrm{Mat}_{n\times n}(\mathbb{C}),\ B, C^T \in \mathbb{C}^n,\ d \in \mathbb{C},$$

then the induced fractional action on $\mathbb{C}^n$

$$\begin{pmatrix} A & B \\ C & d \end{pmatrix} \cdot z = \frac{1}{Cz+d}(Az+B)$$

preserves D_n and ds^2. Again[7] $U(n,1)$ acts transitively on the space of all complex tangent planes, so this metric on D_n has constant holomorphic sectional curvature. Restricting to a totally geodesic surface, which is isometric to H^2, shows that this curvature is negative. The Kähler manifold (D_n, ds^2) is called the *complex hyperbolic space*, denoted $\mathbb{C}H^n$.

Similarly to the real case, the only complete simply connected Kähler manifolds with constant holomorphic sectional curvature are $\mathbb{CP}^n, \mathbb{C}^n$ and $\mathbb{C}H^n$. The proof is also very similar to that in the real case; see Kobayashi and Nomizu, Theorem IX.7.9.

The example of $\mathbb{C}H^n$ suggests a construction of a large class of Kähler metrics. Take a bounded domain $D \subset \mathbb{C}^n$ and define

$$K : D \to \mathbb{R} \;\text{ by }\; K(z) = -\log \mathrm{dist}(z, \partial D).$$

We can try and treat K as a "Kähler potential". In general, $\left[\frac{\partial^2 K}{\partial z_i \partial \bar z_j}\right]$ does not have to be positive definite everywhere, but if it is (such a function K is called a *strictly plurisubharmonic function*), then $i\partial\bar\partial K$ defines a Kähler metric. In this case the domain D is called *strictly*[8] *Levi pseudoconvex* (or just strictly pseudoconvex). In particular, domains which are strictly convex in the usual sense are strictly Levi pseudoconvex, so they carry a natural (complete) Kähler metric.

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[7] Details will be the topic of a homework question.

[8] sometimes "strongly"

Holomorphic sectional curvature of submanifolds

Recall that if M is a smooth submanifold of a Riemannian manifold (N, g), then the Levi-Civita connection of $(M, g|_M)$ is given by the orthogonal projection of the Levi-Civita ∇ connection on N, i.e. if we decompose $\nabla_X Y$, $X, Y \in \Gamma(TM)$, as $(\nabla_X Y)^T + (\nabla_X Y)^\perp$, then the first term is the Levi-Civita connection of M, and the second term is the 2nd fundamental form of M in N, denoted by $\alpha(X, Y)$. The sectional curvatures of M and N are related by the Gauss equation:

$$K_M(X \wedge Y) = K_N(X \wedge Y) + g(\alpha(X, X), \alpha(Y, Y)) - g(\alpha(X, Y), \alpha(X, Y))$$

(here X and Y are orthonormal). Now suppose that (N, g) is Kähler and M is a complex submanifold of N. Then $g|_M$ is Kähler. We have:

Proposition 4.4.2. *The 2nd fundamental form of a complex submanifold M of a Kähler manifold (N, g, J) satisfies*

$$\alpha(JX, Y) = \alpha(X, JY) = J\alpha(X, Y), \quad \forall p \ \forall X, Y \in T_pM.$$

Proof.

$$\alpha(JX, Y) = (\nabla_X JY)^\perp \underset{J \text{ parallel}}{=} (J\nabla_X Y)^\perp \underset{J \text{ orthogonal}}{=} J(\nabla_X Y)^\perp = J\alpha(X, Y).$$

The other equality follows from the symmetry of α in the two arguments. □

We immediately conclude:

Corollary 4.4.3. *The holomorphic sectional curvature of a complex submanifold M of a Kähler manifold (N, g, J) satisfies*

$$K_M(X \wedge JX) = K_N(X \wedge JX) - 2g(\alpha(X, X), \alpha(X, X)), \quad |X| = 1.$$

In particular the holomorphic sectional curvature decreases in submanifolds. □

Observe that there is no statement corresponding to the last one for the sectional curvature of Riemannian manifolds.

Further reading: For more on pseudoconvexity, see Chapter I of Demailly's book, in particular §I.7.

4.5 Kähler quotients

A fundamental construction in Riemannian geometry is that of Riemannian submersions, in particular quotients by a free, proper, and isometric group action. A moment of thought shows that this cannot produce Kähler manifolds from a Kähler manifold: even the dimension of the quotient may be odd. Instead, there exists a different construction, which generalises that of the Fubini-Study metric as S^{2n+1}/S^1.

Let (M, g, J) be a Kähler manifold and let G be Lie group acting holomorphically and isometrically on M. For any element ρ of the Lie algebra $\mathfrak{g}$ of G we have the corresponding *fundamental vector field* X_ρ on M:

$$X_\rho|_m = \frac{d}{ds}\left(e^{s\rho}m\right)\Big|_{s=0}.$$

Since G preserves the Kähler form ω, $L_{X_\rho}\omega = 0$. Hence, using Cartan's magic formula:

$$0 = L_{X_\rho}\omega = \big(di_{X_\rho} + i_{X_\rho}d\big)\omega = di_{X_\rho}\omega,$$

since ω is closed. Therefore the 1-form $i_{X_\rho}\omega$ is closed. We say that X_ρ is *Hamiltonian* if this form is exact, i.e. if there exists a function $\mu^\rho \in C^\infty(M)$ such that $i_{X_\rho}\omega = d\mu^\rho$. Suppose now that every X_ρ is Hamiltonian (e.g. when M is simply-connected). Then we obtain a map $\mu : M \to \mathfrak{g}^*$ given by:

$$\mu(m)(\rho) = \mu^\rho(m).$$

We say that the G-action on M is *Hamiltonian* if the map μ is equivariant, i.e. $\mu(g.m) = g.\mu(m)$, where the action of G on $\mathfrak{g}^*$ is the coadjoint action: $(g.\phi)(\rho) = \phi(\mathrm{Ad}_g\,\rho)$. The map μ is then called a *moment map* for the G-action.

Example 4.5.1. Let $M = \mathbb{C}^{n+1}$ with its standard Kähler structure, and let $G = S^1$ act by the coordinatewise multiplication. The fundamental vector field X_ρ, corresponding to $\rho = it \in \mathfrak{u}(1)$ is simply

$$(itz_0, \dots, itz_n),$$

and hence

$$i_{X_\rho}\omega = i_{X_\rho}\Big(\frac{i}{2}\sum_{k=0}^n dz_k \wedge d\bar{z}_k\Big) = -\frac{1}{2}\sum_{k=0}^n tz_k d\bar{z}_k - \frac{1}{2}\sum_{k=0}^n t\bar{z}_k dz_k = -\frac{1}{2}td\Big(\sum_{k=0}^n |z_k|^2\Big).$$

Therefore the action is Hamiltonian and the moment map is $\mu(z) = \frac{i}{2}|z|^2$ (or $\frac{i}{2}|z|^2 + ic$ for an arbitrary $c \in \mathbb{R}$).

We are going to prove

Theorem 4.5.2. *Let (M, g, J) be a Kähler manifold with an isometric, holomorphic, and Hamiltonian action of a Lie group G, and a moment map $\mu : M \to \mathfrak{g}^*$. Let $c \in \mathfrak{g}^*$ be a fixed point of the coadjoint action and suppose that the action of G on $\mu^{-1}(c)$ is free and proper. Then $\mu^{-1}(c)/G$ is a Kähler manifold, called the* Kähler quotient *of M by G.*

Proof. We need to show two things: that c is a regular value of μ, and that the Kähler structure descends to $\mu^{-1}(c)/G$. Since $d\mu(v)(\rho) = \omega(X_\rho, v)$ and ω is nondegenerate, the kernel of $d\mu$ has dimension $\dim M - \dim\langle X_\rho\rangle_{\rho\in\mathfrak{g}}$. Therefore any point $m \in M$, at which the action is locally free, is a regular point for μ. Consequently $\mu^{-1}(c)/G$ is smooth.

The tangent space to $\mu^{-1}(c)$ consists of vectors v such that $\omega(X_\rho, v) = 0$, i.e. $g(JX_\rho, v) = 0$, for all $\rho \in \mathfrak{g}$. The tangent space to the quotient $\mu^{-1}(c)/G$ at an orbit $G.m$ can be identified with the horizontal subspace in $T_m\mu^{-1}(c)$, i.e. with vectors orthogonal to all X_ρ. Therefore the tangent space to the Kähler quotient $\mu^{-1}(c)/G$ at $G.m$ is identified with the subspace H of T_mM orthogonal to $\langle X_\rho, JX_\rho\rangle_{\rho\in\mathfrak{g}}$. Since J is a pointwise isometry, J acts on H, and, consequently, $\mu^{-1}(c)/G$ is an almost complex manifold. Since the metric on M is Kähler, its Levi-Civita connection ∇ commutes with J. The Levi-Civita connection of $\mu^{-1}(c)/G$ is just the projection of ∇ onto H, and, therefore, it commutes with $J|_H$. Thus the almost complex structure of $\mu^{-1}(c)/G$ is parallel for the Levi-Civita connection, hence integrable, and the induced metric on $\mu^{-1}(c)/G$ is Kähler. ☐

Example 4.5.3. Let us return to Example 4.5.1. Choose $ir \in \mathfrak{g} \simeq \mathfrak{u}(1)$. The set $\mu^{-1}(ir)$ is empty if $r < 0$ and a point if $r = 0$. Therefore the assumptions of the theorem are satisfied only for $r > 0$. In this case $\mu^{-1}(ir)$ is the sphere of radius $2r$ in $\mathbb{C}^n$ and the resulting Kähler metric on $\mu^{-1}(ir)/S^1 \simeq S^{2n+1}/S^1 \simeq \mathbb{CP}^n$ is a constant multiple of the Fubini-Study metric.

Toric Kähler manifolds

We shall now generalise this last example to quotients of a flat $\mathbb{C}^N$ by a torus. Consider the standard torus T^N acting on $\mathbb{C}^N$:

$$\big(e^{it_1}, \dots, e^{it_N}\big).(z_1, \dots, z_N) = \big(e^{it_1}z_1, \dots, e^{it_N}z_N\big),$$

and let S be an $(N-n)$-dimensional subtorus of T^N. If we perform a Kähler quotient of $\mathbb{C}^N$ with respect to S, then the result is a $2n$-dimensional Kähler manifold on which the quotient torus $T^N/S \simeq T^n$ (of half dimension) acts isometrically, holomorphically, and has a moment map (i.e. the T^n-action is Hamiltonian). Such a Kähler manifold is called *toric.*

We shall show that toric Kähler manifolds are in $1-1$ correspondence with certain combinatorial data.

We view S as the kernel of the projection $T^N \to T^n$. Passing to Lie algebras[9], we have an exact sequence

$$0 \longrightarrow \mathfrak{s} \xrightarrow{\iota} \mathbb{R}^N \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0. \tag{4.5.1}$$

Denote by u_i, $i = 1, \dots, N$, the image of the standard generator e_i, i.e. $u_i = \beta(e_i)$. In order to be able to exponentiate this exact sequence (i.e. in order that S is an embedded subtorus) the coordinates of each u_i must be integers. The moment map for T^N is (via a calculation as in Ex. 4.5.1)

$$\mu(z_1, \dots, z_N) = \frac{1}{2}\sum_{k=1}^{N} |z_k|^2 e_k + c.$$

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[9] We now identify the Lie algebra of a torus with $\mathbb{R}^N$, rather than $i\mathbb{R}^N$.

Here we identified the Lie algebra of T^N with its dual using the standard inner product on $\mathbb{R}^N$. The moment map for S is now just the projection of μ onto $\mathfrak{s}^*$, i.e. if we write $\alpha_k = \iota^*(e_k)$, where $\iota^* : \mathbb{R}^N \to \mathfrak{s}^*$ is the projection, then

$$\mu_S(z_1, \dots, z_N) = \frac{1}{2}\sum_{k=1}^{N} |z_k|^2\alpha_k + c. \tag{4.5.2}$$

The constant c is of the form

$$c = \frac{1}{2}\sum_{k=1}^{N} \lambda_k\alpha_k$$

for some scalars $\lambda_1, \dots, \lambda_N \in \mathbb{R}$. If S acts freely on $\mu_S^{-1}(0)$, then $\mu_S^{-1}(0)/S$ is a smooth toric Kähler manifold M. The condition $\mu_S(z) = 0$ means that $\iota^*\bigl(\sum_{k=1}^N(|z_k|^2 + \lambda_k)e_k\bigr) = 0$, i.e. $\sum_{k=1}^N(|z_k|^2 + \lambda_k)e_k \in \operatorname{Ker}\iota^*$. The sequence dual to (4.5.1) implies then that $\sum_{k=1}^N(|z_k|^2 + \lambda_k)e_k \in \operatorname{Im}\beta^*$. Since

$$\beta^*(x) = \sum_{k=1}^{N}\langle x, u_k\rangle e_k,$$

it follows that $z = (z_1, \dots, z_k)$ satisfies $\mu_S(z) = 0$ if and only if there exists an $x \in \mathbb{R}^n$ such that

$$|z_k|^2 + \lambda_k = \langle x, u_k\rangle \;\; \forall\, k = 1, \dots, N.$$

The point $x \in \mathbb{R}^n$ is then the image of $S.z \in M$ under the T^n-moment map on the Kähler quotient M. Consider now the hyperplane H_k given by $\langle x, u_k\rangle = \lambda_k$. Points of M which map to this hyperplane satisfy $z_k = 0$, and hence the circle e^{it_k} acts trivially at those points. Therefore the hyperplanes H_k are the images of fixed point sets of circles in T^n. It follows also that such a fixed point set has (real) codimension 2 in M and locally $M \simeq X \times \mathbb{R}^2$, where S^1 acts trivially on X and in a standard way on $\mathbb{R}^2$. The moment map μ_{S^1} for the circle action is then just the moment map for the action on $\mathbb{R}^2 \simeq \mathbb{C}$, i.e. $\frac{1}{2}|z|^2$. It follows that μ_{S^1} maps M to one side of the hyperplane H_k, and consequently the image of the moment map for the T^n action on M is the intersection of half-spaces

$$\langle x, u_k\rangle \geq \lambda_k, \;\; k = 1, \dots, N.$$

Observe that such an intersection of half-spaces determines all the data needed to perform a Kähler quotient: since we know the vectors u_k we can determine the subtorus S from (4.5.1), and since we know the constants λ_k, we know c in (4.5.2). Therefore we can recover M from its image in $\mathbb{R}^n$. Of course, if M is to be a manifold, then the hyperplanes H_k must satisfy certain conditions. I shall just state them here and leave a proof as an exercise or to look up.

Proposition 4.5.4. *The Kähler quotient $\mu_S^{-1}(0)/S$ constructed above is smooth if and only if whenever m hyperplanes $H_{k_1}, \dots, H_{k_m}$ have a nonempty intersection, then their normal vectors $u_{k_1}, \dots, u_{k_m}$ are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ (recall that the u_k have integer coordinates). In particular, at most n hyperplanes can have a nonempty intersection.* $\square$

In particular, compact toric Kähler manifolds of dimension $2n$ are obtained from convex polytopes in $\mathbb{R}^n$, the supporting hyperplanes of which satisfy this condition. Such polytopes are called *Delzant polytopes.*

Example 4.5.5. Consider the standard simplex Δ in $\mathbb{R}^n$ with vertices at the origin and the points $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$. The normal vectors to the faces of Δ are $u_1 = e_1, \dots, u_n = e_n, u_{n+1} = -e_1 - \cdots - e_n$. In particular, it is a Delzant polytope. The subtorus S of T^{n+1} determined by (4.5.1) has the Lie algebra $\{(t_1, \dots, t_{n+1}); \sum t_k u_k = 0\}$, i.e. $t_1 = t_2 = \cdots = t_{n+1}$. The scalars λ_k are $0, \dots, 0, -1$, and the moment map μ_S is $\frac{1}{2}\big(|z|^2 - 1\big)$. Therefore (Ex. 4.5.1) the resulting toric Kähler manifold is $\mathbb{CP}^n$ with its Fubini-Study metric.

Further reading: For more on (compact) toric Kähler manifolds see the two (very well written) Appendices in the book "*Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*" by V. Guillemin (Birkhäuser 1994).
For a beautiful introduction to toric geometry (as part of algebraic geometry) see "*Introduction to toric varieties*" by W. Fulton (Princeton University Press, 1993).

Chapter 5

Calabi-Yau and Kähler-Einstein

This chapter is about Ricci curvature of Kähler manifolds, in particular about finding Kähler metrics with "best" Ricci curvature.

5.1 Ricci curvature of Kähler manifolds

Recall that we have defined the Ricci form of a connection D on a complex vector bundle as $\operatorname{tr} R^D$ and showed that it is a closed 2-form. Suppose now that $E \overset{\pi}{\to} M$ is a holomorphic vector bundle over a complex manifold and D is the Chern connection of a hermitian metric h on E. Then in a local holomorphic frame $\{e_1, \dots, e_k\}$:

$$\operatorname{tr} R^D = -\partial\bar{\partial} \log \det[h_{ij}], \quad \text{where} \quad h_{ij} = \langle e_i, e_j \rangle = h(e_i, e_j).$$

Let now (M, J, g) be a Kähler manifold. Then we have two Ricci curvatures on the tangent bundle of M. On the one hand we have the the above Ricci form on $E = TM$, where D is the Chern connection of the Kähler metric. We shall usually make the Ricci form real:

Definition 5.1.1. The **Ricci form** ρ of a Kähler manifold is defined as $i \operatorname{tr} R^D$. It is a real closed (1,1)-form.

On the other hand, one defines the Ricci curvature of any Riemannian metric:

$$\operatorname{Ric}(X, Y) = \operatorname{tr}(V \longmapsto R^{\nabla}(V, X)Y)$$

where ∇ is the Levi-Civita connection. It is a symmetric $(2, 0)$-tensor. Equivalently we can define a $(1, 1)$-tensor

$$\operatorname{Ric}: \ TM \to TM, \quad g(\operatorname{Ric}(X), Y) = \operatorname{Ric}(X, Y).$$

The two objects are related as follows:

Proposition 5.1.2. *On a Kähler manifold*

$$\operatorname{Ric}(X,Y)=\rho(X,JY).$$

Proof. Recall the first Bianchi identity for any torsion-free linear connection:

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0.$$

Hence

$$\begin{aligned}\operatorname{Ric}(X,Y)&=\operatorname{Ric}(Y,X)=\operatorname{tr}\left(V\longmapsto R(V,Y)X\right)=\operatorname{tr}\left(V\longmapsto -JR(V,Y)JX\right)\\&=\operatorname{tr}(V\longmapsto (JR(Y,JX)V+JR(JX,V)Y)).\end{aligned}$$

On the other hand:

$$\begin{aligned}-\operatorname{Ric}(X,Y)&=-\operatorname{tr}(V\longmapsto R(V,X)Y)=\operatorname{tr}(V\longmapsto R(X,V)Y)\\&=\operatorname{tr}(JV\longmapsto JR(X,V)Y)\underset{\substack{\text{tr is a}\\ \text{(1,1)-form}}}{=}\operatorname{tr}(JV\longmapsto JR(JX,JV)Y)\\&\underset{JV\mapsto V}{=}\operatorname{tr}(V\longmapsto JR(JX,V)Y).\end{aligned}$$

Therefore $\operatorname{Ric}(X,Y)=\operatorname{tr}(V\longmapsto JR(Y,JX)V)-\operatorname{Ric}(X,Y)$ and so

$$\operatorname{Ric}(X,Y)=\frac{1}{2}\operatorname{tr}(V\longmapsto JR(Y,JX)V).$$

Now choose a local orthonormal frame of TM of the form $E_1,JE_1,E_2,JE_2,\dots,E_n,JE_n$. The last formula can be rewritten as

$$\begin{aligned}\operatorname{Ric}(X,Y)&=\frac{1}{2}\sum_i g(JR(Y,JX)E_i,E_i)+\frac{1}{2}\sum_i g(JR(Y,JX)JE_i,JE_i)\\&=\sum_i g(JR(Y,JX)E_i,E_i).\end{aligned}$$

On the other hand we can compute the trace of the curvature of the Chern connection with respect to the hermitian metric

$$h=\sum g_{pq}\;dz_p\otimes d\overline{z}_q=g-i\omega.$$

Since $E_1,\dots,E_n$ is a unitary frame of TM, we have

$$\begin{aligned}\operatorname{tr}R(X,Y)&=\sum_i h(R(X,Y)E_i,E_i)=\sqrt{-1}\sum_i\omega(R(X,Y)E_i,E_i)\\&=-\sqrt{-1}\sum_i g(JR(X,Y)E_i,E_i).\end{aligned}$$

Hence $\operatorname{Ric}(X,Y)=-i\operatorname{tr}R(Y,JX)=-\rho(Y,JX)=\rho(JY,X)$. □

It follows (cf. p. 64) that we have the following simple formula for the Ricci curvature of a Kähler manifold in local complex coordinates

$$\mathrm{Ric} = -\frac{1}{2}\,\mathrm{Re}\sum_{p,q}\frac{\partial^2 \log\det[g_{pq}]}{\partial z_p\partial\bar{z}_q}\;dz_p d\bar{z}_q.$$

Corollary 5.1.3. *The Ricci curvature of a Kähler metric depends only on the complex structure and on the volume form of the metric.* □

Now, if we change the Kähler metric from g to g', then the volume form changes from ω^n to $e^f\omega^n$ for a real function f and the Ricci form will change to

$$\rho' = \rho - i\partial\bar\partial f.$$

In particular, the Ricci form of a Kähler metric varies in a fixed cohomology class, which of course is $2\pi c_1(TM) = 2\pi c_1(M)$.

We may ask, as we did earlier for the hermitian vector bundles, whether any real closed $(1,1)$-form φ with $[\varphi] = 2\pi c_1(M)$ is the Ricci-form of a Kähler metric? Equivalently, is any volume form μ the volume form of a Kähler metric? This is the famous *Calabi problem.* The answer is yes, if M is compact (Yau, 1978).

Remark 5.1.4. Observe that we already know that we can find a *hermitian metric* with prescribed Ricci curvature on any compact Kähler manifold. Indeed, we established in §3.4 that this is true on any manifold on which the global $\partial\bar\partial$-lemma holds, i.e. any real exact $(1,1)$-form is of the form $i\partial\bar\partial f$. In the last homework you showed that this lemma holds on any compact Kähler manifold. Of course the problem finding a *Kähler metric* with prescribed Ricci curvature is much harder.

Another natural condition on a Kähler metric is the Einstein equation: $\mathrm{Ric} = \lambda g$ for some constant λ (often expressed as "Ricci curvature is constant"). On a Kähler manifold we can write this as

$$i\,\mathrm{tr}\,R^\nabla = \rho = \lambda\omega.$$

Such metrics are called *Kähler-Einstein.* Observe that a metric with constant holomorphic sectional curvature has constant Ricci curvature, so we have first examples of Kähler-Einstein manifolds: $\mathbb{CP}^n, \mathbb{C}^n, \mathbb{C}H^n$, with their standard metrics.

Example 5.1.5. Let $M = G/H$ be a compact *homogeneous* Kähler manifold (e.g. projective spaces, Grassmannians, or flag manifolds). On such a manifold there is only one (up to a constant multiple) G-invariant volume form (this is the volume form of the normal metric, discussed in the "Riemannian Geometry" course). Hence, owing to Corollary 5.1.3, the Ricci form of any G-invariant Kähler metric is a fixed real $(1,1)$-form ρ. If one shows that ρ is positive definite, then by taking ρ as the fundamental form of a Kähler metric, one can conclude that M admits a unique (up to a constant multiple) G-invariant Kähler-Einstein metric (with positive Einstein constant). This Ricci form ρ is indeed positive definite, but a proof of this requires a substantial detour into Lie theory. See Chapter 8 in Besse's *"Einstein manifolds* (Springer, 1987).

5.2 Calabi-Yau theorem

We have seen that the first Chern class of a Kähler manifold is represented by $\frac{1}{2\pi}\rho$, where ρ is the Ricci form defined in the previous section. The following question is known as the *Calabi problem*:

Let M be a complex manifold. Is any closed real $(1,1)$-form φ with $[\varphi] = 2\pi c_1(M)$ the Ricci form of a Kähler metric?

Yau showed in 1977 that the answer is yes if M is compact (after presenting a (wrong) counterexample in 1973):

Theorem 5.2.1 (Calabi-Yau theorem)**.** *Let M be a compact complex manifold which admits of a Kähler metric g with Kähler form ω. Any closed real $(1,1)$-form φ with $[\varphi] = 2\pi c_1(M)$ is the Ricci form of a unique Kähler metric $\widetilde{g}$ in the same Kähler class as ω (i.e. $[\omega] = [\widetilde{\omega}]$).*

Corollary 5.2.2. *If M is a compact Kähler manifold with $c_1(M) = 0$, then M admits a Ricci-flat Kähler metric.*

Example 5.2.3. We have seen in Example 3.4.10 that the K3-surface

$$S = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

has vanishing first Chern class. Therefore S admits a Ricci-flat Kähler metric (which is unknown explicitly).

We are going to discuss a proof of the Calabi-Yau theorem. We have seen that the Ricci form of a Kähler metric depends only on the volume form of the metric (once we fix the complex structure). If we change the Kähler metric $g \longmapsto g'$, then the volume form changes by a conformal factor

$$\omega^n \longmapsto (\omega')^n = e^f \omega^n$$

for some $f \in C^\infty(M)$ and the Ricci form changes as

$$\rho \longmapsto \rho' = \rho - i\partial\bar{\partial} f.$$

Now, if $[\varphi] = [\rho]$, then the global $\partial\bar{\partial}$-lemma implies that there exists an f such that $\rho - \varphi = i\partial\bar{\partial} f$. Moreover, any two such f, f' differ by a constant (on each connected component), since their difference is harmonic. We can fix this constant by requiring that

$$\int\limits_M e^f \omega^n = \int\limits_M \omega^n.$$

Observe that this last condition is automatically satisfied by any f such that $(\omega')^n = e^f \omega^n$ and $[\omega'] = [\omega]$. We therefore have an equivalent formulation of the Calabi-Yau theorem:

The map $\omega' \longmapsto \log \frac{(\omega')^n}{\omega^n}$ from the space of Kähler metrics in the Kähler class $[\omega]$ to the set

$$\Big\{ f \in C^\infty(M) \; ; \; \int\limits_M e^f \omega^n = \int\limits_M \omega^n \Big\}$$

is a bijection.

Let us now reinterpret the domain of this map: Since $[\omega'] = [\omega]$, the global $\partial\bar{\partial}$-lemma implies that $\omega' - \omega = i\partial\bar{\partial}u$ for some real function u. Again, any two such functions differ by a constant, which we can fix by requiring that

$$\int\limits_M u\,\omega^n = 0.$$

Thus we have two spaces of smooth functions:

$$\mathcal{K} = \Big\{u \in C^\infty(M)\ ;\ \omega + i\partial\bar{\partial}u > 0,\ \int\limits_M u\,\omega^n = 0\Big\}$$
$$\mathcal{K}' = \Big\{f \in C^\infty(M)\ ;\ \int\limits_M e^f\omega^n = \int\limits_M \omega^n\Big\}$$

and a map $\mathrm{Cal} : \mathcal{K} \to \mathcal{K}'$ given by

$$\mathrm{Cal}(u) = \log\frac{(\omega + i\partial\bar{\partial}u)^n}{\omega^n}.$$

The Calabi-Yau theorem says that Cal is a bijection (or even a diffeomorphism if we view $\mathcal{K}, \mathcal{K}'$ as ∞-dimensional manifolds). In local complex coordinates, if the given metric g is written as

$$g = \sum g_{p\overline{q}}\ dz_p d\overline{z}_q,$$

then the map Cal is

$$\mathrm{Cal}(u) = \log\det\left[g_{p\overline{q}} + \frac{\partial^2 u}{\partial z_p \partial\overline{z}_q}\right] - \log\det\left[g_{p\overline{q}}\right].$$

This is an example of a *complex Monge-Ampère equation*[1]. It is highly nonlinear, but it is a single equation (unlike the general Riemannian Einstein equations). The simplest complex Monge-Ampère equation is

$$\det\left[\frac{\partial^2 u}{\partial z_p \partial\overline{z}_q}\right] = h(z_p, \overline{z}_q)$$

for some (positive) function h on $\mathbb{C}^n$. Finding a plurisubhamonic solution u, i.e. one such that hermitian matrix $\left[\frac{\partial^2 u}{\partial z_p \partial\overline{z}_q}\right]$ is positive-definite, means that we have found a Kähler metric on $\mathbb{C}^n$ with Ricci form $i\partial\bar{\partial}\log h$ (u is a Kähler potential of this metric). In particular, if h is constant, then a plurisubharmonic solution

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[1] A (real or complex) Monge-Ampère equation is a second order PDE which involves the determinant of the matrix of second derivatives. In 1781 Monge wanted to move “rubble” in order to build a fortification, while minimising the cost. The problem can be expressed as a real Monge-Ampère equation.

gives a Ricci-flat Kähler metric on $\mathbb{C}^n$ (or on its domain of definition). The solution $u(z) = c\sum |z_p|^2$ gives the standard flat metric.

Returning to the proof of the Calabi-Yau theorem, we begin by showing that the map Cal is injective (proved by Calabi in 1955):

Proposition 5.2.4. *Let M be a compact complex manifold. The map* Cal : $\mathcal{K} \to \mathcal{K}'$ *is injective.*

Proof. Suppose that ω_1 and $\omega_2 = \omega_1 + i\partial\bar{\partial}u$ have the same volume form. Since forms of even degree commute, we have

$$0 = \omega_2^n - \omega_1^n = (\omega_2 - \omega_1) \wedge \sum_{k=0}^{n-1} \omega_1^k \wedge \omega_2^{n-k-1} = i\partial\bar{\partial}u \wedge \sum_{k=0}^{n-1} \omega_1^k \wedge \omega_2^{n-k-1}$$

for some $u \in \mathcal{K}$. The form $\sigma = \sum\limits_{k=0}^{n-1} \omega_1^k \wedge \omega_2^{n-k-1}$ is an $(n-1, n-1)$-form, which in local coordinates can be written as

$$\sum M^{p\overline{q}}\, dz_1 \wedge \cdots \wedge \widehat{dz_p} \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge \widehat{d\overline{z}_q} \wedge \cdots \wedge dz_n,$$

so that our equation becomes

$$0 = \sum M^{p\overline{q}} \frac{\partial^2 u}{\partial z_p \partial \overline{z}_q}.$$

Let us multiply the equation $0 = i\partial\bar{\partial}u \wedge \sigma$ by $2u$ and use the identity $2i\partial\bar{\partial} = dd^c$, where $d^c = i(\bar{\partial} - \partial)$, to obtain:

$$0 = 2iu\partial\bar{\partial}u \wedge \sigma = udd^c u \wedge \sigma = d(ud^c u \wedge \sigma) - du \wedge d^c u \wedge \sigma.$$

Integrating yields

$$0 = \int\limits_M du \wedge d^c u \wedge \sigma.$$

Since $\partial u = du + iJdu$, we get

$$0 = \int\limits_M du \wedge Jdu \wedge \sigma.$$

Since ω_1 defines a Kähler metric, we can find a local frame of the form $\{e_1, Je_1, \ldots e_n, Je_n\}$ such that

$$\omega_1 = \sum_{j=1}^n e_j \wedge Je_j, \quad \omega_2 = \sum_{j=1}^n a_j e_j \wedge Je_j,$$

where a_j are strictly positive local functions. It follows that

$$\omega_1^k \wedge \omega_2^{n-k-1} = * \left(\sum_{j=1}^n b_{jk} e_j \wedge Je_j \right)$$

where $b_{jk} = \text{(factorials)} \sum\limits_{\substack{j_1<\dots<j_k \\ j_s\neq j}} a_{j_1}\dots a_{j_k} > 0$. If

$$du = \sum \alpha_i e_i + \sum \beta_i J e_i, \quad \text{then} \quad Jdu = \sum \alpha_i J e_i - \sum \beta_i e_i, \quad \text{and}$$

$$du \wedge Jdu \wedge \sigma = \langle du \wedge Jdu, *\sigma \rangle\ \omega_1^n = \Big(\sum_{j,k} (\alpha_j^2 + \beta_j^2) b_{jk}\Big)\omega_1^n.$$

Hence the integrand is positive and the equation $0 = \int\limits_M du \wedge Jdu \wedge \sigma$ implies that $\alpha_j = \beta_j = 0$, i.e. $du = 0$, so u is constant, and therefore $u = 0$ since $\int\limits_M u\, \omega_1^n = 0$. ☐

We now turn to the surjectivity of Cal, i.e. to the existence of solutions to the Monge–Ampère equation. We need the following simple lemma:

Lemma 5.2.5. *Let g be a Kähler metric, given in local coordinates by*

$$g = \sum g_{p\bar{q}}\ dz_p d\bar{z}_q.$$

Then the $\bar\partial$-Laplacian $\Delta_{\bar\partial} = \frac{1}{2}\Delta_d$ is given on functions by

$$\Delta_{\bar\partial}\ u = -\sum g^{p\bar{q}} \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}, \quad \text{where } \left[g^{p\bar{q}}\right] = \left[g_{p\bar{q}}\right]^{-1}.$$

In particular, the Laplacian on Kähler manifolds does not depend on the derivatives of the metric tensor.

Proof. The formula is evidently true on $\mathbb{C}^n$, where $g_{p\bar{q}} = \delta_{p\bar{q}}$. Therefore it is true in Kähler normal coordinates at any point. The right-hand side can, however, be written as

$$- * (i\partial\bar\partial u \wedge \omega^{n-1}),$$

which means that the identity is independent of the choice of coordinates. ☐

Returning to the surjectivity of Cal, we observe, first of all, that any solution u to our Monge–Ampère equation automatically belongs to $\mathcal{K}$, i.e. $\omega + i\partial\bar\partial u$ defines a Kähler metric. In local coordinates this means that $\left[g_{p\bar{q}} + \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}\right]$ is positive definite. This is obviously true at a point p_0, where u attains a local minimum. Suppose that there exists a point p_1 at which one of the eigenvalues is nonpositive. This means that on the path from p_1 to p_0 there is a point at which one of the eigenvalues is zero. But this contradicts the Monge-Ampère equation, which can be rewritten (in local coordinates near p_1) as:

$$\det\left[g_{p\bar{q}} + \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}\right] \det[g_{p\bar{q}}]^{-1} = e^f > 0.$$

Thus the problem of positivity of the metric is out of the way and we "only" need to solve $\mathrm{Cal}(u) = f$ for some given f. For this one uses the so-called *continuity method.* We consider the set

$$I(f) = \{t \in [0,1] \ ; \ \mathrm{Cal}(u) = tf \text{ has a solution}\}.$$

Since $\mathrm{Cal}(0) = 0$, $0 \in I(f)$. We need to show that $I(f)$ is open and closed. First of all, we need to decide on a Banach space in which we seek solutions. These are the Hölder spaces $C^{2,\alpha}(M)$, $\alpha \in (0,1)$. Recall that the $C^{0,\alpha}$-semi-norm of a function is

$$\|\varphi\|_{0,\alpha}' = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{\alpha}},$$

and $\|\varphi\|_{k,\alpha} = \|\varphi\|_{C^k} + \max_{|\lambda|=k} \left\|D^{\lambda}\varphi\right\|_{0,\alpha}'$.

Lemma 5.2.6. *Let u be a $C^{2,\alpha}$-solution of* $\mathrm{Cal}(u) = f$, *where f is smooth. Then u is smooth.*

Proof. In local complex coordinates the equation is:

$$\log\det\left[g_{p\bar{q}} + \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}\right] - \log\det\left[g_{p\bar{q}}\right] = f.$$

Differentiate this with respect to any local coordinate x, and get:

$$\mathrm{tr}\left(\partial_x \left[g_{p\bar{q}} + \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}\right]\left[g_{p\bar{q}} + \frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}\right]^{-1}\right) = \text{something smooth}$$

After swapping the matrices under tr, and assuming that $u \in C^{k,\alpha}$, $k \geq 2$, this can be written as:

$$-\frac{1}{2}\Delta_{g_u}(\partial_x u) + (\text{something in } C^{k-2,\alpha}) = \text{smooth}.$$

Δ_{g_u} is a second order elliptic operator with $C^{k-2,\alpha}$-bounded coefficients (owing to Lemma 5.2.5). The usual Schauder estimates[2] imply now that $\partial_x u \in C^{k,\alpha}$, i.e. $u \in C^{k+1,\alpha}$, and repeating shows $u \in C^{\infty}(M)$. □

Let us show that $I(f)$ is open. Write $\omega_u = \omega + i\partial\bar{\partial}u$ and compute the differential of the map Cal:

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} \log\frac{(\omega + i\partial\bar{\partial}(u+\varepsilon v))^n}{\omega_u^n} = n\frac{i\partial\bar{\partial}v \wedge \omega_u^{n-1}}{\omega_u^n} = -\frac{1}{2}\Delta_{g_u} v.$$

Since the Laplacian is an isomorphism between $C^{k+2,\alpha}$ and $C^{k,\alpha}$, a version of the inverse function theorem implies that Cal is an open maping and hence $I(f)$ is open.

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[2] See, e.g., D. Gilbarg and N.S. Trudinger "*Elliptic Partial Differential Equations of Second Order*", Springer (1983, 2001).

It remains to show that $I(f)$ is closed, or, equivalently, that Cal is a proper mapping. Let $t_n \in I(f)$ and $t_n \to t \in [0,1]$ and let u_n be the corresponding unique smooth solutions of $\mathrm{Cal}(u) = t_n f$. If $\alpha > \beta$, then the inclusion $C^{2,\alpha}(M) \hookrightarrow C^{2,\beta}(M)$ is compact (i.e. the image of a bounded set is relatively compact) - this is similar to usual Arzelà-Ascoli theorem. Thus as soon as we have a uniform estimate of the $C^{2,\alpha}$-norm of the u_n for some α, then (u_n) has a convergent subsequence in $C^{2,\beta}(M)$, $\beta < \alpha$, and hence $t \in I(f)$.

Therefore one needs a priori $C^{2,\alpha}$-estimates on solutions. About this part, which is of course the key to the proof, I shall say only a few words. C^0-estimates are very hard, and due to Yau. C^2-estimates are obtained by differentiating $\mathrm{Cal}(u) = f$ twice as above, and getting an estimate on $\Delta_g u$. C^3-estimates are then obtained by the following trick, which is due to Calabi. Consider the positive function:

$$S = \sum g_u^{\alpha\bar{\lambda}} g_u^{\mu\bar{\beta}} g^{\gamma\bar{\nu}} \left(\partial_{\alpha\bar{\beta}\gamma} u\right) \left(\partial_{\bar{\lambda}\mu\bar{\nu}} u\right).$$

The Laplace equation for S, together with C^2-estimates on u, gives an estimate on S. Since the C^2-estimates are estimates on g_u, i.e. on the coefficients on S, this yields C^3-estimates on u.

Remark 5.2.7. It is now known that this last trick is actually a property of equations of Monge-Ampère type: the $C^{2,\alpha}$-estimates follow from the C^2-estimates (see Gilbarg and Trudinger, op. cit., Theorem 17.14).

Further reading:

(i) For all the analytic details of the proof, see (apart from Yau's original paper) Chapter 5 of D. Joyce's book "*Compact manifolds with special holonomy*" (OUP 2000).

(ii) As we have seen, not every compact manifold manifold admits a Kähler metric. On the other hand, as mentioned at the end of §4.2, every compact complex manifold admits a *Gauduchon metric*, i.e. one where $\partial\bar{\partial}\omega^{n-1} = 0$. Motivated by the Calabi-Yau theorem, Gauduchon asked in 1984 whether we can prescribe the Ricci curvature of a Gauduchon metric. This has been proved in 2017 by G. Székelyhidi, V. Tosatti, and B. Weinkove, *"Gauduchon metrics with prescribed volume form"*, Acta Math., 219 (2017), 181–211. It is perhaps worth pointing out that, since the global $\partial\bar{\partial}$-lemma does not need to hold on M, one seeks a metric with the Ricci form in the same *Bott-Chern* cohomology class as $2\pi c_1(M)$.

5.3 Aubin-Yau theorem

The Calabi-Yau theorem implies, in particular, that on any compact Kähler manifold M with $c_1(M) = 0$ there exists a Ricci-flat Kähler metric. This solves the problem of existence of Kähler-Einstein metrics with zero Einstein constant, but what about nonzero Einstein constant, i.e. Kähler metrics which satisfy $\rho = \lambda\omega$, $\lambda \neq 0$? Clearly we require $c_1(M) > 0$ or $c_1(M) < 0$ (as defined in Remark 4.3.10). The Calabi-Yau theorem gives only a weaker conclusion:

Corollary 5.3.1. *If M is a compact complex manifold with $c_1(M) > 0$ (resp. $c_1(M) < 0$), then M admits a Kähler metric with positive (resp. negative) Ricci curvature.*

Remark 5.3.2. Observe that if $c_1(M) > 0$ or $c_1(M) < 0$, then M admits a Kähler metric.

The case of $c_1(M) < 0$ has been completely answered by Aubin and Yau (independently):

Theorem 5.3.3 (Aubin-Yau)**.** *Let M be a compact complex manifold with negative first Chern class. Then M has a unique (up to rescaling) Kähler-Einstein metric (with negative Einstein constant).*

Example 5.3.4. Let $M \subset \mathbb{CP}^n$ be the smooth zero set of a homogeneous polynomial of degree d. The same computation as for the K3-surface (Ex. 3.4.10) shows that

$$c_1(M) = (n+1-d)\, c_1(\mathbb{CP}^n)|_M.$$

Therefore, provided $d > n+1$, such an M carries a Kähler-Einstein metric with $\lambda < 0$.

Sketch of a proof. Choose $\lambda < 0$. Since $c_1(M) < 0$, there exists a positive real $(1,1)$-form ω such that $\lambda\omega \in 2\pi c_1(M)$. In particular $g(X,Y) = \omega(X, JY)$ is a Kähler metric. We seek a Kähler-Einstein metric g' with Kähler form ω', Ricci-form ρ' and Einstein constant λ, i.e. $\rho' = \lambda\omega'$. Since $\rho' \in 2\pi c_1(M)$, $\lambda\omega' \in 2\pi c_1(M)$ and so, once again, $[\omega] = [\omega']$ means that

$$\omega' - \omega = i\partial\bar{\partial}u \iff \rho' - \rho = -i\partial\bar{\partial}\,\mathrm{Cal}(u).$$

Since $[\rho] = [\lambda\omega]$, there exists a smooth function f, unique up to an additive constant, such that $\rho - \lambda\omega = i\partial\bar{\partial}f$, and this can be rewritten as

$$\rho' - \rho = \lambda\omega' - \lambda\omega - i\partial\bar{\partial}f = \lambda i\partial\bar{\partial}u - i\partial\bar{\partial}f.$$

Therefore $-i\partial\bar{\partial}\,\mathrm{Cal}(u) = -i\partial\bar{\partial}(f - \lambda u)$. Thus we need to show this time that there exists a unique solution to the equation $\mathrm{Cal}(u) = -\lambda u + f$ for a given f, where $u \in \mathcal{K}$ and $\lambda < 0$. This is a different Monge-Ampère equation. The positivity of $\omega + i\partial\bar{\partial}u$ follows as before, as do the uniqueness and the regularity. For the existence on uses again the continuity method, i.e. one considers the set

$$I(f) = \{t \in [0,1]\,;\ \exists u : \mathrm{Cal}(u) = -\lambda u + tf\}.$$

Again $0 \in I(f)$ ($u = 0$ is a solution). $I(f)$ is open, since the linearisation of the equation is

$$-\frac{1}{2}\Delta_{g_u} v + \lambda v = 0$$

and the Laplacian does not have negative eigenvalues:

$$0 > \int\limits_M 2\lambda v^2 \omega_u^n = \int\limits_M \langle \Delta v, v\rangle \omega_u^n = \int\limits_M \langle d^* dv, v\rangle \omega_u^n = \int\limits_M \langle dv, dv\rangle \omega_u^n \geq 0,$$

which is a contradiction. Therefore the linearised map is again an isomorphism and the inverse function theorem implies that $\mathrm{Cal}(u) + \lambda u$ is an open mapping.

For the remainder (i.e. the closedness of $I(f)$) one needs estimates similar to those in the Calabi-Yau theorem. This time, however, they are easier to obtain (and the Aubin-Yau theorem was proved before Theorem 5.2.1). In particular, a C^0-estimate is very easy: we have a solution of

$$\log\bigl(\omega + i\partial\bar{\partial}u\bigr)^n - \log\omega^n = -\lambda u + f,$$

which means that at a maximum (resp. minimum) of u we have $-\lambda u + f \leq 0$ (resp. $-\lambda u + f \geq 0$). Therefore, at any $m \in M$, $|u(m)| \leq |\lambda|^{-1}\sup_M |f|$. $\square$

Remark 5.3.5 (Extremal metrics). Kähler-Einstein metrics can exist only if $c_1(M)$ is positive, negative, or zero, and then only in this cohomology class. But suppose we choose an arbitrary cohomology class $\Omega \in H^2_{\mathrm{dR}}(M)$. What would be the "best" Kähler metric in this class (i.e. with $[\omega] = \Omega$)?

One option is to look for metrics with constant *scalar* curvature[3] S. More generally, one looks for critical points of the functional (also called *Calabi functional*)

$$\Omega \ni \omega \longmapsto \int_M S(\omega)^2 \omega^n. \tag{5.3.1}$$

Such a metric is called *extremal.* One can show that a Kähler metric is extremal if and only if the $(1,0)$-part of the (Riemannian) gradient of the scalar curvature is a holomorphic vector field. This means that on a compact complex manifold, which does not have global holomorphic vector fields (i.e. $H^0(M, T^{1,0}M) = 0$), any extremal metric has constant scalar curvature. In general, there do exist extremal metrics with non-constant scalar curvature. There also exist projective manifolds without any extremal metrics.

Further reading: For more on extremal metrics, see §11.E in Besse's *"Einstein manifolds"*, or *"An Introduction to Extremal Kähler Metrics"* by G. Székelyhidi (AMS, 2014).

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[3] Scalar curvature is the trace of the Ricci curvature.

5.4 Obstructions in the case $c_1(M) > 0$

In the remaining case, $c_1(M) > 0$, any attempt to prove Theorem 5.3.3 encounters problem after problem. Certainly the uniqueness statement fails: consider the Fubini-Study metric g on $\mathbb{CP}^n$ which is Kähler-Einstein. Let ϕ be a notrivial biholomorphism of $\mathbb{CP}^n$ which is not an isometry, i.e. $\phi \in PGL(n+1, \mathbb{C}) \backslash PU(n+1)$. Then $\phi^* g$ is a Kähler-Einstein metric different from g.

The openness of $I(f)$ cannot be proved in the same way as for $c_1 \leq 0$. This is not critical; Aubin has shown how to overcome this. Instead of solving $\mathrm{Cal}(u) = -\lambda u + tf$, we consider the equation $\mathrm{Cal}(u) = -t\lambda u + f$. Calabi-Yau theorem implies that this has a solution for $t = 0$. If u is a solution for some $t \geq 0$, and we set $\omega_t = \omega + i\partial\bar{\partial}u$, then the Ricci form of ω_t is

$$\rho_t = \lambda t \omega_t + \lambda(1-t)\omega_0.$$

Therefore the Ricci curvature is greater than λt if $t < 1$. The linearised operator is $-\frac{1}{2}\Delta_{g_t} + t\lambda$. We can now appeal to a result of Lichnerowicz (proved in the next subsection), who showed that the first nonzero eigenvalue λ_1 of the Laplacian on a compact Kähler manifold with Ricci $> \mu > 0$ satisfies $\lambda_1 \geq 2\mu$. Hence, in our case, the linearised operator is invertible for any $t \in [0, 1)$ for which a solution exists. This is enough for the continuity method to work, provided we can show that $I(f)$ is closed. This, as explained above, requires apriori estimates. The C^2- and C^3-estimates do not depend on the sign of λ and continue to hold. However, the C^0-estimate might fail! For a good reason, too: there are compact Kähler manifolds with $c_1 > 0$, which do not admit Kähler-Einstein metrics.

The obstructions, as we shall now see, have to do with automorphic (i.e. real-holomorphic) transformations of Kähler manifolds.

Killing vector fields on compact Kähler-Einstein manifolds

Recall that a *Killing vector field* X on a Riemannian manifold (M, g) is the same as an infinitesimal isometry, i.e. $L_X g = 0$. On the other hand, a real-holomorphic vector field (Definition 1.5.11) is an infinitesimal automorphism of the complex structure, i.e. $L_X J = 0$. In other words X is the real part of a global holomorphic vector field.

As we shall now see, on compact Kähler-Einstein manifolds real holomorphic and Killing vector fields are closely related. First of all, we have the following application of the Hodge-de Rham theorem:

Theorem 5.4.1. *An infinitesimal isometry on a compact Kähler manifold is also an infinitesimal automorphism of the complex structure. In other words, a continuous group of isometries preserves the complex structure as well.*

Proof. Let $\phi_t : M \to M$ be a continuous 1-parameter group of isometries, $t \in I$, $\phi_0 = \mathrm{Id}$, i.e. ϕ_t is obtained by integrating tX, where X is a Killing vector field. Let ω be the fundamental form of the Kähler metric, i.e. $\omega(v, w) = g(Jv, w)$. We need to show that $\phi_t^* \omega = \omega$. On a Kähler manifold, ω is parallel, hence closed and co-closed, hence harmonic. Since each ϕ_t is an isometry, it commutes with the Hodge star, so it takes harmonic forms to harmonic forms. The Hodge-de Rham theorem implies that we have the commutative diagram

$$\begin{tikzcd}
H^k(M) \arrow[r, "{[\phi_t]}"] \arrow[d, "\simeq"'] & H^k(M) \arrow[d, "\simeq"'] \\
\mathcal{H}^k_\Delta(M) \arrow[r, "\phi_t^*"] & \mathcal{H}^k_\Delta(M).
\end{tikzcd}$$

The upper horizontal map is the identity since each ϕ_t is homotopic to identity. Therefore the lower horizontal map is also the identity. □

Remark 5.4.2. Observe the difference with the non-compact case: the Euclidean metric on $\mathbb{C}^n$ is preserved by all $A \in SO(2n, \mathbb{R})$, but the complex structure only by $A \in U(n) \subset SO(2n, \mathbb{R})$. Observe also that the above is statement false for discrete groups of isometries, e.g. antipodal map on $S^2 \simeq \mathbb{C}P^1$.

We are now going to prove (Δ denotes the Riemannian Laplacian):

Theorem 5.4.3 (Matsushima). *Let M be a compact Kähler-Einstein manifold with nonzero Einstein constant λ. Then the Killing vector fields are in $1-1$ correspondence with functions u such that $\Delta u = 2\lambda u$. In particular, if $\lambda < 0$, then there are no Killing vector fields on M, i.e. the isometry group of M is discrete.*

Proof. First of all, the second statement follows from the first by integration:

$$2\lambda \int\limits_M u^2 = \int\limits_M u\Delta u = \int\limits_M \langle du, du \rangle.$$

For the first part, we need:

Lemma 5.4.4. *Let X be a real-holomorphic vector field on a Kähler manifold. For any smooth function f we have:*

$$2i(X)\left(\sqrt{-1}\partial\bar{\partial} f\right) = Jd(X(f)) + d((JX)(f)).$$

Proof. First of all $2\sqrt{-1}\partial\bar{\partial} = dJd$. Using Cartan's magic formula $L_X = di(X) + i(X)d$ for differential forms, we obtain:

$$i(X)dJdf = L_X(Jdf) - di(X)Jdf = JL_X(df) + di(JX)df = Jd(X(f)) + dL_{JX}f.$$

□

Proof of the theorem: Suppose that X is a Killing vector field (hence automorphic, due to Theorem 5.4.1) and apply this lemma to the local function $f = \ln\det[g_{ij}]$. We obtain

$$-2i(X)\rho = dL_{JX}\ln\det[g_{ij}]$$

since X is Killing. Now $L_{JX}\omega^n = h\omega^n$ for some $h \in C^\infty(M)$, which means that $L_{JX} \ln \det[g_{ij}] = h$. Hence $i(X)\rho$ is exact, and since $i(X)\rho = \lambda i(X)\omega$ and $\lambda \neq 0$, $i(X)\omega$ is exact. We can therefore write $i(X)\omega = du$ for some $u \in C^\infty(M)$, which means that $\operatorname{grad} u = JX$ $(du(Y) = \omega(X,Y) = g(JX,Y))$. On the other hand, for any $f \in C^\infty(M)$, $L_{\operatorname{grad} f}\omega^n = -(\Delta f)\omega^n$. Therefore

$$h\omega^n = L_{JX}\omega^n = L_{\operatorname{grad} u}\omega^n = -(\Delta u)\omega^n,$$

which means that $h = -\Delta u$. Consequently:

$$d\Delta u = -dh = 2i(X)\rho = 2\lambda i(X)\omega = 2\lambda du,$$

i.e. $d(\Delta u - 2\lambda u) = 0$ and $\Delta u - 2\lambda u$ is constant on each connected component. Since u is defined only up to an additive constant and $\lambda \neq 0$, we have exactly one u such that $\Delta u = 2\lambda u$.

For the other direction we need the aforementioned result of Lichnerowicz:

Theorem 5.4.5 (Lichnerowicz)**.** *Let M be a compact Kähler manifold with Ricci curvature $\geq \lambda > 0$. Then the first nonzero eigenvalue λ_1 of Δ satisfies $\lambda_1 \geq 2\lambda$. Equality implies that the gradient vector field $X = \operatorname{grad}\varphi$ of any eigenfunction φ for λ is automorphic and satisfies $\operatorname{Ric}(X) = \lambda X$.*

Before proving this, let us see how Theorem 5.4.3 follows. Let $u \in C^\infty(M)$ satisfy $\Delta u = 2\lambda u$. Theorem 5.4.5 implies that $X = \operatorname{grad} u$ is automorphic and $\operatorname{Ric}(X) = \lambda X$. Hence, for any vector field Y,

$$\rho(JX,Y) = -\rho(X,JY) = -\operatorname{Ric}(X,Y) = -g(\operatorname{Ricci}(X),Y) = -\lambda g(X,Y) = -\lambda du(Y),$$

and so $i(JX)\rho = -\lambda du$, i.e. $i(JX)\omega = -du$, i.e. $L_{JX}\omega = 0$. But we also have $L_{JX}J = JL_X J = 0$, and hence JX is an infinitesimal isometry.

It remains to prove the Lichnerowicz theorem. Let X be a vector field on a Kähler manifold and consider ∇X as an endomorphism of the tangent bundle $z \mapsto \nabla_z X$. We can decompose:

$$\nabla X = \nabla^{1,0}X + \nabla^{0,1}X = \frac{1}{2}\left(\nabla X - J \circ \nabla X \circ J\right) + \frac{1}{2}\left(\nabla X + J \circ \nabla X \circ J\right).$$

X is automorphic if and only if $\nabla X \circ J = J \circ \nabla X$, i.e. $\nabla^{0,1}X = 0$. We now compute

$$\nabla^*(J \circ \nabla X \circ J) = J \circ \nabla^*(\nabla X \circ J) = -\operatorname{Ric}(X),$$

since $\operatorname{Ric}(X) = \sum R(E_i, JE_i)JX$ for a local frame $\{E_1, JE_1, \ldots, E_n, JE_n\}$. Therefore

$$\nabla^*\nabla X = \frac{1}{2}(\nabla^*\nabla X + \operatorname{Ric}(X)) + \nabla^*\nabla^{0,1}X,$$

which is equivalent to

$$2\nabla^*\nabla^{0,1}X = \nabla^*\nabla X - \operatorname{Ric}(X). \tag{5.4.1}$$

We need one more ingredient from Riemannian geometry: the Bochner identity[4] says that on a Riemannian manifold

$$\left(\Delta X^\flat\right)^\sharp = \nabla^*\nabla X + \mathrm{Ric}(X).$$

If φ is an eigenfunction of the Laplacian, i.e. $\Delta\varphi = \mu\varphi$, then $X = \operatorname{grad}\varphi$ satisfies

$$\left(\Delta X^\flat\right)^\sharp = (\Delta d\varphi)^\sharp = (d\Delta\varphi)^\sharp = \mu(d\varphi)^\sharp = \mu X.$$

The Bochner formula yields $\mu X = \nabla^*\nabla X + \mathrm{Ric}(X)$, which we can rewrite as

$$\nabla^*\nabla X = (\mu - 2\lambda)X + (2\lambda X - \mathrm{Ric}(X)).$$

Formula 5.4.1 gives now:

$$\nabla^*\nabla^{0,1} X = \frac{1}{2}(\mu - 2\lambda)X + (\lambda X - \mathrm{Ric}(X)).$$

Therefore $\mathrm{Ric}(X) \geq \lambda$ implies

$$0 \leq \underbrace{\left\|\nabla^{0,1}X\right\|_2^2}_{L^2-\text{norm}} = \langle\nabla^{0,1}X, \nabla^{0,1}X\rangle_{L^2} = \langle\nabla^{0,1}X, \nabla X\rangle = \langle\nabla^*\nabla^{0,1}X, X\rangle$$
$$= \frac{1}{2}(\mu - 2\lambda)\|X\|_2^2 + \langle\lambda X - \mathrm{Ric}(X), X\rangle \leq \frac{1}{2}(\mu - 2\lambda)\|X\|_2^2,$$

and hence $X = \operatorname{grad}\varphi$ is nonzero only if $\mu \geq 2\lambda$. Moreover the equality is equivalent to $\nabla^{0,1}(X) = 0$ and $\lambda X = \mathrm{Ric}(X)$.

This finishes the proof of the Lichnerowicz theorem, and hence also the proof of Theorem 5.4.3. We have the following important application:

Theorem 5.4.6 (Matsushima)**.** *Let M be a compact Kähler-Einstein manifold with positive Einstein constant. Then any infinitesimal automorphism X of the complex structure is of the form $X = X_1 + JX_2$, where X_1 and X_2 are Killing vector fields.*

Proof. Recall the formula from Lemma 5.4.4:

$$2i(X)\sqrt{-1}\partial\bar{\partial}f = Jd(X(f)) + d((JX)(f)), \quad \forall f \in C^\infty(M).$$

In particular, applying this to $f = -\ln\det[g_{ij}]$ yields

$$2i(X)\rho = Jdh_1 + dh_2,$$

where

$$h_1 = X(-\ln\det[g_{ij}]) = -\frac{X(\det[g_{ij}])}{\det[g_{ij}]}, \quad h_2 = JX(-\ln\det[g_{ij}]).$$

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[4] See, e.g., P. Petersen *"Riemannian Geometry"* (Springer, 2006), Corollary 7.21 (p. 216).

Therefore $h_1\omega^n = -L_X\omega^n$ and $h_2\omega^n = -L_{JX}\omega^n$. Since $\rho = \lambda\omega$, we get $2i(X)\omega = Jd\left(\frac{h_1}{\lambda}\right) + d\left(\frac{h_2}{\lambda}\right)$. Since ω is nondegenerate, there exist vector fields Y_1 and Y_2 such that

$$i(Y_1)\omega = Jd\left(\frac{h_1}{2\lambda}\right), \quad i(Y_2)\omega = d\left(\frac{h_2}{2\lambda}\right),$$

so that $X = Y_1 + Y_2$. It follows that $L_{Y_2}\omega = 0$ and $L_{JY_1}\omega = 0$. Now

$$i(JY_2)g = i(Y_2)\omega = d\left(\frac{h_2}{\lambda}\right) \iff \operatorname{grad}\left(\frac{h_2}{2\lambda}\right) = JY_2.$$

On the other hand

$$h_2\omega^n = -L_{JX}\omega^n \underset{L_{JY_1}\omega^n=0}{=} -L_{JY_2}\omega^n = \Delta\left(\frac{h_2}{2\lambda}\right)\omega^n.$$

Therefore $\Delta h_2 = 2\lambda h_2$ and Theorem 5.4.3 implies that Y_2 is a Killing vector field. The same argument shows that JY_1 is Killing. □

This gives the following restriction on the group of biholomorphisms of compact Kähler-Einstein manifolds:

Corollary 5.4.7. *Let M be a compact complex manifold. If M admits a Kähler-Einstein metric with $\lambda > 0$, then the Lie algebra of the group of biholomorphisms of M is* reductive, *i.e. the complexification of the Lie algebra of a compact Lie group.*

Proof. Follows immediately from the fact that the isometry group of a compact Riemannian manifold is compact. □

Remark 5.4.8. The conclusion of this corollary holds already for compact Kähler manifolds with constant scalar curvature. This is due to Lichnerowicz; see Besse's *"Einstein manifolds"*, Proposition 2.151.

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The Futaki invariant

Another obtstruction to existence of Kähler-Einstein metrics with $\lambda > 0$ is given by the so-called *Futaki invariant*, which is a linear functional on the space $\mathfrak{a}(M)$ of real-holomorphic vector fields (recall that $\mathfrak{a}(M) \simeq H^0(M, T^{1,0}M)$). First of all define, on any compact Kähler manifold, a *Ricci potential* to be a function F such that $\rho - i\partial\bar{\partial}F$ is harmonic. Such an F exists: let ν be the unique harmonic form in $[\rho] = c_1(L)$; then $\rho - \nu$ is an exact real $(1,1)$-form, and hence of the form $i\partial\bar{\partial}F$ (due to the global $\partial\bar{\partial}$-lemma). We can make F unique by requiring that its integral over M vanishes. The Futaki invariant is defined as

$$\mathcal{F}(X) = \int_M X(F)\omega^n, \quad X \in \mathfrak{a}(M).$$

Proposition 5.4.9. *Let (M, g) be a compact Kähler manifold. If the scalar curvature of g is constant, then the Ricci potential F is identically zero. Consequently, the Futaki invariant vanishes as well.*

Proof. From the definition of F, $\rho = \phi + i\partial\bar{\partial}F$, where ϕ is harmonic. Therefore the scalar curvature S satisfies $\frac{1}{2}S = \operatorname{tr}\rho = \operatorname{tr}\phi + \frac{1}{2}\Delta F$. Since ϕ is harmonic, its trace is constant, so $S = \text{const}$ implies that ΔF is constant, hence equal to zero, and so F must be zero. ☐

Remark 5.4.10. The definition of the Futaki invariant may seem strange at first sight. One reason for interest is that $\mathcal{F}$ actually depends only on the cohomology class of ω and not on the metric itself (see Corollary 2.160 in Besse's *"Einstein manifolds"*). Even more importantly, it turned out that a generalisation of the Futaki invariant, due to Donaldson, is precisely what one needs in order to characterise compact complex manifolds with $c_1 > 0$ which admit a Kähler-Einstein metric.

5.5 Blowing-up and examples with no Kähler-Einstein metric

We are going to give an example of a compact Kähler manifold with $c_1 > 0$ which does not satisfy the conclusion of Corollary 5.4.7 and therefore has no Kähler-Einstein metric. In order to do this, we need one of the most important constructions in complex geometry[5] - blowing up a point or, more generally, a subvariety.

We begin with the local construction. Let Δ be a disk in $\mathbb{C}^n$, centred at the origin and define

$$\widetilde{\Delta} = \left\{(z,l) \in \Delta \times \mathbb{CP}^{n-1};\, z \in l\right\} = \left\{\big((z_1,\dots,z_n),[l_1,\dots,l_n]\big);\, z_i l_j = z_j l_i\right\}.$$

We have the projection

$$\pi : \widetilde{\Delta} \to \Delta, \quad \pi(z,l) = z.$$

If $z \neq 0$, then there is a unique line through 0 containing z, so that π is an isomorphism away from $0 \in \Delta$. On the other hand $\pi^{-1}(0) = \mathbb{CP}^{n-1}$. The manifold $\widetilde{\Delta}$ with the projection onto Δ is called the *blow-up* of Δ at 0. Observe that it effectively separates all lines passing through 0; one should think of it as parametrising points of Δ *and* tangent directions at 0.

Observe also that for $\Delta = \mathbb{C}^n$ the projection onto the other factor, $\widetilde{\mathbb{C}^n} \to \mathbb{CP}^{n-1}$, identifies $\widetilde{\mathbb{C}^n}$ with the tautological line bundle J on $\mathbb{CP}^{n-1}$.

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[5] Blowing-up is becoming increasingly important in real differential geometry. For example, it is used to describe the asymptotic behaviour of large classes of complete Riemannian metrics.

Let now M be a complex manifold of dimension n and U a neighbourhood of $x \in M$ biholomorphic to Δ. We define the blow-up $\widetilde{M}_x$ of M at x to be the complex manifold obtained by replacing U with its blow-up $\tilde{U} \simeq \widetilde{\Delta}$, i.e.:

$$\widetilde{M}_x = (M \backslash U) \sqcup \widetilde{U}.$$

Again there is a projection $\pi : \widetilde{M}_x \to M$ which is an isomorphism away from x. The inverse image $E_x = \pi^{-1}(x) \simeq \mathbb{CP}^{n-1}$ is called the *exceptional divisor* of the blow-up.

Remark 5.5.1. More generally, we can blow up a complex submanifold Y of M by replacing Y with the projectivisation of its normal bundle $N_Y = TM|_Y/TY$. Intuitively, we separate all normal directions at every point of Y.

We are going to compute the first Chern class of a blow-up at a point. We compute in the second cohomology group. We can decompose $\widetilde{M} = \widetilde{M}_x$ as the union of $\widetilde{M} \backslash E \simeq \pi^{-1}(M \backslash \{x\})$ and a tubular neighbourhood W of E in $\widetilde{M}$. W is isomorphic to a neighbourhood of the zero section in the normal bundle of E. We may assume that $n = \dim_{\mathbb{C}} M \geq 2$, since blowing up is a trivial operation in dimension 1. The Mayer-Vietoris sequence implies then that $c_1(M) = c_1(M \backslash \{x\})$. On the other hand $c_1(\widetilde{M} \backslash E) \simeq \pi^* c_1(M \backslash \{x\})$ (since π is an isomorphism outside x), and hence $c_1(\widetilde{M} \backslash E) = \pi^* c_1(M)$. Since $W \backslash E$ is isomorphic to punctured disk and $n \geq 2$, $W \backslash E$ has no cohomology in dimension 1 or 2. Therefore the Mayer-Vietoris sequence implies that $c_1(\widetilde{M}) = \pi^* c_1(M) + c_1(W)$. We need to compute $c_1(W) = c_1(TW)$. Since W can be deformed to E, we only need to compute $c_1(TW\big|_E)$. Since the normal bundle of E is isomorphic to the tautological bundle J on $\mathbb{CP}^{n-1}$, the projection $W \to E$ induces an exact sequence

$$0 \to J \to TW\big|_E \to TE \to 0,$$

where J is the tautological bundle on $E \simeq \mathbb{CP}^{n-1}$. Therefore

$$\Lambda^n \big(TW\big|_E\big) \simeq J \otimes \Lambda^{n-1} TE = J \otimes K^*_{\mathbb{CP}^{n-1}} \simeq J \otimes H^n \simeq H^{n-1}.$$

Thus, finally:

$$c_1(\widetilde{M}) = \pi^* c_1(M) + (n-1) c_1(H).$$

We can identify the class $c_1(H)$ as follows:

Lemma 5.5.2. *The line bundle J on E is isomorphic to $[E]|_E$.*[6] *Consequently, $c_1(H) = -\eta_E$, where η_E is the Poincaré dual of E, and*

$$c_1(\widetilde{M}) = \pi^* c_1(M) - (n-1)\eta_E. \tag{5.5.1}$$

Proof. The first statement is local, so we can assume that $M = \Delta$. Consider the pullback of $J = \mathcal{O}(-1)$ from $\mathbb{CP}^{n-1}$ under the second projection $\widetilde{\Delta} \to \mathbb{CP}^{n-1}$. This bundle has a section $s : (z, l) \mapsto \big((z, l), z\big)$, which vanishes along E. This proves the first statement, and the second one follows from Theorem 3.5.7. □

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[6] Recall the definition of a line bundle corresponding to a divisor from §3.5.

Del Pezzo surfaces

We shall now investigate the positivity of the first Chern class of the blow-up of $\mathbb{CP}^2$ at several points. First of all, we have:

Proposition 5.5.3. *Let S be the blow-up of $\mathbb{CP}^2$ at one or two points. Then $c_1(S) > 0$.*

Proof. Consider first the blow-up at one point, say $p = [1, 0, 0]$. In terms of local coordinates $[z_0, z_1, z_2]$ the local coordinates near p are $z_1/z_0, z_2/z_0$ and the definition of the blow-up given above means that we can describe the blow-up $S = \widetilde{\mathbb{CP}^2}_p$ as a hypersurface in $\mathbb{CP}^2 \times \mathbb{CP}^1$:

$$\big\{([z], [w]) \in \mathbb{CP}^2 \times \mathbb{CP}^1;\ z_1 w_1 - z_2 w_0 = 0\big\}.$$

We have shown in (3.4.2) that $K_S^* \simeq K_M^*\big|_M \otimes N_S^*$, where $M = \mathbb{CP}^2 \times \mathbb{CP}^1$ and N_S is the normal bundle of S in M. Let π_1, π_2 be the projections from $\mathbb{CP}^2 \times \mathbb{CP}^1$ onto the two factors. Then $TM \simeq \pi_1^* T\mathbb{CP}^2 \oplus \pi_2^* T\mathbb{CP}^1$ and taking the exterior powers show that $K_M^* \simeq \pi_1^* \mathcal{O}_{\mathbb{CP}^2}(3) \otimes \pi_2^* \mathcal{O}_{\mathbb{CP}^1}(2)$. On the other hand, since S is defined by an equation of degree 1 in z and of degree 1 in w, $N_S \simeq \big(\pi_1^* \mathcal{O}_{\mathbb{CP}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{CP}^1}(1)\big)\big|_S$. Combining these formulae yields:

$$K_S^* \simeq \big(\pi_1^* \mathcal{O}_{\mathbb{CP}^2}(2) \otimes \pi_2^* \mathcal{O}_{\mathbb{CP}^1}(1)\big)\big|_S.$$

Both $\mathcal{O}_{\mathbb{CP}^2}(2)$ and $\mathcal{O}_{\mathbb{CP}^1}(1)$ have hermitian metrics with positive Ricci form. The Ricci form of the tensor metric on $\big(\pi_1^* \mathcal{O}_{\mathbb{CP}^2}(2) \otimes \pi_2^* \mathcal{O}_{\mathbb{CP}^1}(1)\big)$ is just the sum of the two Ricci forms, so it is positive, and of course it remains positive when restricted to S. Thus $c_1(S) > 0$.

For two points, we can similarly describe the blow-up S as submanifold of $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ cut out by two equations of degrees $(1, 1, 0)$ and $(1, 0, 1)$. A similar computation gives now

$$K_S^* \simeq \big(\pi_1^* \mathcal{O}_{\mathbb{CP}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{CP}^1}(1) \otimes \pi_3^* \mathcal{O}_{\mathbb{CP}^1}(1)\big)\big|_S,$$

and again we can conclude that $c_1(S) > 0$. ☐

The argument in the proof clearly breaks down for a blow-up at 3 points. Nevertheless one can show (although we shall not do this) that the blow-up of $\mathbb{CP}^2$ at three points still satisfies $c_1 > 0$, provided these three points do not lie on a line (i.e. $\mathbb{CP}^1$) in $\mathbb{CP}^2$. So can we get 2-dimensional projective manifolds with $c_1 > 0$ by blowing up $\mathbb{CP}^2$ at an arbitrary number of points? The answer is no, as we shall now see.

Clearly if a cohomology class $\alpha \in H^{1,1}(M)$ is positive then $\int_Y \alpha^m > 0$ for any m-dimensional submanifold (or subvariety) Y of M.[7] We are going to compute $\int_S c_1(S)^2$ for blow-ups of $\mathbb{CP}^2$.

We need a preparatory result:

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[7] Remarkably, if $\alpha = c_1(L)$, then the converse is also true. This is known as the *Nakai-Moishezon criterion.*

Lemma 5.5.4. *Let E be the exceptional divisor of the blow-up of $\mathbb{CP}^2$ at a point. Then the self-intersection number of E is -1, i.e. $\eta_E \cap \eta_E = -1$.*

Proof. Without loss of generality we may assume that we blow up $x = [1, 0, 0]$. Consider the meromorphic function $f = z_1/z_0$ on $\mathbb{CP}^2$ and its composition $\tilde{f} = f \circ \pi$ with $\pi : \widetilde{\mathbb{CP}^2} \to \mathbb{CP}^2$. The divisor of $\tilde{f}$ is $E + H_1 - H_0$, where H_0, H_1 are the pullbacks of hyperplanes $\{z_0 = 0\}$, $\{z_1 = 0\}$. This means that the line bundles $[E]$ and $[H_0 - H_1]$ are isomorphic, and hence, owing to Theorem 3.5.7, $\eta_E = \eta_{H_0} - \eta_{H_1}$. Now observe, directly from the definition of the blow-up, that E does not intersect H_0 and it intersects H_1 in one point. Hence $E.E = E.(H_0 - H_1) = -1$. □

Remark 5.5.5. Since blow-up is a local construction, this lemma is valid for any complex surface S. The fact that $E.E = -1$ means that we cannot move E inside $\tilde{S}_p$. Indeed, if we could, then E and its deformation E' would intersect in one point, but with opposite orientations. This means that E' is not a complex submanifold of $\tilde{S}_p$.

Let us now blow up $\mathbb{CP}^2$ at k distinct points $x_1, \ldots, x_k$. We know from (5.5.1) that

$$c_1(\widetilde{\mathbb{CP}^2}_{x_1,\ldots,x_k}) = c_1(\mathbb{CP}^2) - \eta_{E_1} - \cdots - \eta_{E_k} = 3c_1(H) - \eta_{E_1} - \cdots - \eta_{E_k}.$$

Therefore (where we identify highest cohomology with $\mathbb{C}$ via integration)

$$c_1(\widetilde{\mathbb{CP}^2}_{x_1,\ldots,x_k})^2 = (3\underbrace{c_1(H)}_{=\eta_{\mathbb{CP}^1}} - \sum_i \eta_{E_i})^2 = 9\underbrace{\eta^2_{\mathbb{CP}^1}}_{=1} - 6\sum_i \underbrace{\eta_{\mathbb{CP}^1} \cdot \eta_{E_i}}_{=0} + \sum_i \underbrace{\eta^2_{E_i}}_{=-1} = 9 - k.$$

Thus $c_1\left(\widetilde{\mathbb{CP}^2}_{x_1,\ldots,x_k}\right)$ can be positive only if $k \leq 8$. It turns out that for a generic choice of up 8 points, the first Chern class of this surface is indeed positive. These manifolds are known as *del Pezzo surfaces*[8].

We shall now relate the group of automorphisms (i.e. biholomorphisms) of a surface to that of its blow-up.

Proposition 5.5.6. *Let S be a complex surface and $\widetilde{S}_p$ the blow-up of S at $p \in S$. Then the connected component of identity $\mathrm{Aut}_0(\widetilde{S}_p)$ of the group of automorphisms of $\widetilde{S}_p$ is isomorphic to the stabiliser of p in $\mathrm{Aut}_0(S)$:*

$$\mathrm{Aut}_0(\widetilde{S}_p) \simeq \mathrm{Aut}_0(S, p) = \big\{\Phi \in \mathrm{Aut}_0(S)\ ;\ \Phi(p) = p\big\}.$$

Proof. If $\Psi \in \mathrm{Aut}(\widetilde{S}_p)$, then $\Psi^*\eta_E \cdot \Psi^*\eta_E = -1$. Therefore Ψ maps E to another curve with self-intersection -1. If Ψ is close to identity then, as explained in Remark 5.5.5, this curve must be E, so that $\mathrm{Aut}_0(\widetilde{S}_p)$ preserves E, and hence preserves $\widetilde{S}_p \backslash E$. The restriction of Ψ to $\widetilde{S}_p \backslash E$ defines an element of $\mathrm{Aut}_0(S, p)$. Conversely, an automorphism $\Phi \in \mathrm{Aut}_0(M, p)$ defines an automorphism $\tilde{\Phi}$ of $\widetilde{S}_p$ by setting $\tilde{\Phi}|_{\widetilde{S}_p \backslash E} = \Phi_{S\backslash\{p\}}$ and $\tilde{\Phi}|_E = d\Phi_p$ (recall that $E \simeq \mathbb{P}(T_pS)$). □

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[8] A del Pezzo surface is, by definition, a 2-dimensional projective manifold with $c_1 > 0$. The only one which is not a blow-up of $\mathbb{CP}^2$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$.

We can finally give examples of compact complex manifolds with $c_1(M) > 0$ and no Kähler-Einstein metric.

Example 5.5.7. Consider the blow-up of $\mathbb{CP}^2$ in one or two points, say $p_1 = [1, 0, 0]$ in the first case, and $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$ in the second case. We know from Proposition 5.5.3 that these surfaces have positive first Chern class. Proposition 5.5.6 implies that

$$\mathrm{Aut}_0\Big(\widetilde{\mathbb{CP}^2}_{p_1}\Big) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in PGL(3, \mathbb{C}) \right\},$$

$$\mathrm{Aut}_0\Big(\widetilde{\mathbb{CP}^2}_{p_1,p_2}\Big) = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in PGL(3, \mathbb{C}) \right\}.$$

These groups are nonreductive, and therefore Corollary 5.4.7 implies that $\widetilde{\mathbb{CP}^2}_{p_1}$, $\widetilde{\mathbb{CP}^2}_{p_1,p_2}$ do not admit Kähler-Einstein metrics.

Remark 5.5.8. For 3 points, the corresponding group will be nonreductive only if the 3 points are collinear. But then, as remarked after Proposition 5.5.3, the first Chern class of the blow-up is not positive. In fact, all other del Pezzo surfaces (i.e. apart from blow-ups of $\mathbb{CP}^2$ at one or two points) do carry a Kähler-Einstein metric. This has been shown by G. Tian, *"On Calabi's conjecture for complex surfaces with positive first Chern class"*, Invent. Math. 101 (1990), 101–172. For a readable exposition of the proof, see V. Tosatti, *"Kähler-Einstein metrics on Fano surfaces"*, Expo. Math. (30) (2012), 11–31.

Remark 5.5.9. It is now known which projective manifolds with $c_1 > 0$ admit a Kähler-Einstein metric. For this, one needs to consider the Futaki invariant not just of a given manifold M, but of its possible degenerations, i.e. complex spaces $\mathcal{M}$ with a morphism $\mathcal{M} \to \mathbb{C}$ such the fibres $\mathcal{M}_z$ over $z \neq 0$ are all isomorphic to M (but at $z = 0$ nasty things can happen to M). The space $\mathcal{M}$ has to satisfy certain conditions; in particular, there is also a line bundle $\mathcal{L}$ over $\mathcal{M}$, which restricted to fibres over $z \neq 0$ is isomorphic to K_M^*, and the pair $(\mathcal{M}, \mathcal{L})$ is equipped with an action of $\mathbb{C}^*$ covering the standard action of $\mathbb{C}^*$ on $\mathbb{C}$.[9] One can then define the Futaki invariant $\mathcal{F}_0$ of the central fibre $\mathcal{M}_0$, and M is said to be *K-polystable* if $\mathcal{F}_0 \geq 0$ for all such deformations $\mathcal{M}$ with equality if and only if $\mathcal{M} \simeq M \times \mathbb{C}$. In 2012 X.X. Chen, S. Donaldson and S. Sun proved that a projective manifold M with $c_1(M) > 0$ admits a Kähler-Einstein metric if and only if M is K-polystable. This result is the culmination of almost 60 years of efforts by many famous mathematicians.

An analogous characterisation of manifolds admitting a constant scalar curvature Kähler metric, or, more generally, an extremal metric, is still unknown. It should also be related to K-stability, but a precise formulation, not to mention a proof, is still unclear.

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[9] The remaining condition is that the projection $\mathcal{M} \to \mathbb{C}$ is *flat*. I shall not attempt to explain what this means, since "for every geometric description of flatness there is a counterexample". It does guarantee, however, that the degenerations are reasonably well-behaved.

Chapter 6

Kodaira embedding theorem

Every projective manifold is Kähler, but not conversely. This chapter is concerned first of all with characterising compact Kähler manifolds which can be embedded into a projective space, and, later, with properties of projective manifolds.

6.1 Line bundles and maps into projective spaces

Let M be a compact complex manifold and L a line bundle on M, such that $\dim H^0(M,L) \geq 2$, i.e. L has at least two linearly independent global sections. To every point $p \in M$ we associate the subspace H_p of global sections which vanish at p. We have two possibilities: either $H_p = H^0(M,L)$ or H_p has codimension one. We want to exclude the first possibility: a line bundle is called *base-point free*[1] if for every $p \in M$ there exists a global section $s \in H^0(M,L)$ such that $s(p) \neq 0$.

If a line bundle is base-point free then to every point $p \in M$ we can associate a hyperplane $H_p = \{s \in H^0(M,L);\ s(p) = 0\}$. A hyperplane in a vector space V is the same as a line in the dual space V^*, and therefore we obtain a holomorphic map

$$\Phi_L : M \to \mathbb{P}\big(H^0(M,L)^*\big), \quad p \mapsto H_p,$$

to a projective space of dimension $\dim H^0(M,L) - 1$. Equivalently (but less canonically) we can choose a basis $s_0, \dots, s_N$ of $H^0(M,L)$, and on any open subset $U \subset M$ where L is trivial with a local frame e and $s_i(x) = f_i(x)e$, set

$$\Phi_L(x) = [f_0(x), \dots, f_N(x)] \in \mathbb{CP}^N.$$

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[1] The base locus of a line bundle, or, more generally, of a linear subspace $V \subset H^0(M,L)$, is the set $\{p \in M; s(p) = 0\ \ \forall s \in V\}$.

Remark 6.1.1. Observe that the bundle L is the pullback of the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{P}\big(H^0(M,L)^*\big)$: $L \simeq \Phi_L^*\mathcal{O}(1)$.

Example 6.1.2 (Veronese embedding). We consider $M = \mathbb{CP}^n$ and $L = \mathcal{O}(d) = H^d$. Global sections of $\mathcal{O}(d)$ are homogeneous polynomials of degree d in $n+1$ variables $z_0, \dots, z_n$. It is clear that the base locus is empty. The map $\Phi_{\mathcal{O}(d)} : \mathbb{CP}^n \to \mathbb{CP}^N$, where $N = \dim H^0(\mathbb{CP}^n, \mathcal{O}(d)) - 1 = \binom{n+d}{n} - 1$, is called the *Veronese embedding.*

We can describe it more explicitly by choosing a basis of $H^0(\mathbb{CP}^n, \mathcal{O}(d))$ consisting of all monomials $z^{\underline{\alpha}} = z_0^{\alpha_0} \dots z_n^{\alpha_n}$ with $\alpha_1, \dots, \alpha_n \geq 0$ and $\sum \alpha_j = d$. Let $\underline{\alpha}_0, \dots, \underline{\alpha}_N$ be some ordering of these monomials. Then

$$\Phi_{\mathcal{O}(d)}\big([z_0, \dots, z_n]\big) = \big[z^{\underline{\alpha}_0}, \dots, z^{\underline{\alpha}_N}\big].$$

The image $\mathcal{V}_{n,d}$ of the Veronese embedding, a *Veronese variety*, is the zero locus of the obvious quadratic equations in $\mathbb{CP}^N$: if $\underline{\alpha}_i, \underline{\alpha}_j, \underline{\alpha}_k, \underline{\alpha}_l$ is a quadruple of multi-indices such that $z^{\underline{\alpha}_i} z^{\underline{\alpha}_j} = z^{\underline{\alpha}_k} z^{\underline{\alpha}_l}$, then $\mathcal{V}_{n,d}$ lies on the quadric $w_{\underline{\alpha}_i} w_{\underline{\alpha}_j} = w_{\underline{\alpha}_k} w_{\underline{\alpha}_l}$ in $\mathbb{CP}^N$. For example, $\mathcal{V}_{1,d}$ (which is called a *rational normal curve*) is a submanifold of $\mathbb{CP}^d$ described by vanishing of all 2×2 minors of

$$\begin{bmatrix} w_0 & w_1 & \dots & w_{d-1} \\ w_1 & w_2 & \dots & w_d \end{bmatrix},$$

where $w_0, \dots, w_d$ are homogeneous coordinates on $\mathbb{CP}^d$. For $d = 2$ we get a single equation $w_0 w_2 = w_1^2$.

In the above example, we have not actually proved that the Veronese embedding is an embedding (although this particular case is easy to prove directly), so let us address this question for a general basepoint-free line bundle L and the map Φ_L. A holomorphic (or smooth) map is an embedding if it is: (i) injective, and (ii) an immersion. For the map Φ_L injectivity means that for every pair x, y of distinct points in M we can find a section $s \in H^0(M, L)$ such that $s(x) = 0$ and $s(y) \neq 0$. One says then that the line bundle L *separates points.* We can also express this property by saying that the natural map

$$H^0(M, L) \longrightarrow L_x \oplus L_y,$$

given by evaluating sections at x and y, is surjective for every $x \neq y$. Incidentally, the basepoint-free property can be expressed similarly: the natural map $H^0(M, L) \longrightarrow L_x$ is surjective for every $x \in M$ (one says that L is *generated by sections*). These evaluations maps can be also viewed as arising from long exact cohomology sequences associated to exact sequence of sheaves:

$$0 \longrightarrow L \otimes \mathcal{I}_x \longrightarrow L \longrightarrow L_x \longrightarrow 0, \tag{6.1.1}$$

$$0 \longrightarrow L \otimes \mathcal{I}_{x,y} \longrightarrow L \longrightarrow L_x \oplus L_y \longrightarrow 0, \tag{6.1.2}$$

where $\mathcal{I}_x$ (resp. $\mathcal{I}_{x,y}$) is the (ideal) sheaf of holomorphic functions vanishing at x (resp. vanishing at x and y). The map Φ_L associates $H^0(M, L \otimes \mathcal{I}_x)$ to x.

The second condition, that of immersion, means that the differential of Φ_L is injective at every $x \in M$. In other words, for every $v \in T_xM$, there is a section s of L vanishing at x, but such that "$ds(v)$"$\neq 0$. I claim that ds is well defined as an element of $(T^*M \otimes L)_x$. Indeed, choose any local trivialisation near x, so that the section s is represented by a smooth map $s_0 : U \times \mathbb{C}$ and define ds in the usual way (as the differential of a smooth map). In any other trivialisation, s is represented by $s_1 = gs_0$, where g is the change of trivialisations. Then $ds_1|_x = (s_0dg)|_x + (gds_0)|_x = g(x)ds_0|_x$, since $s_0(x) = 0$, and consequently $ds_0|_x$ represents an element of $(T^*M \otimes L)_x$. Therefore, condition (ii) can be rephrased as: the well defined sheaf map

$$d_x : L \otimes \mathcal{I}_x \to T^*_xM \otimes L_x$$

is surjective on global sections for every $x \in M$. One says that the line bundle L *separates tangent directions*. The sheaf map d_x also fits into a short exact sequence:

$$0 \longrightarrow L \otimes \mathcal{I}_x^2 \longrightarrow L \otimes \mathcal{I}_x \xrightarrow{d_x} T^*_xM \otimes L_x \longrightarrow 0. \tag{6.1.3}$$

Remark 6.1.3. Observe that $\mathcal{I}_x/\mathcal{I}_x^2$ is canonically isomorphic to T^*_xM. Indeed, T^*_xM can be viewed as $\mathcal{H}^{1,0}(M)/\mathcal{H}^{1,0}(M) \otimes \mathcal{I}_x$ (quotient of the sheaf of holomorphic 1-forms by 1-forms vanishing at x), and the differential d_x maps $\mathcal{I}_x^2$ (sheaf of holomorphic functions vanishing to order 2 at x) to $\mathcal{H}^{1,0}(M) \otimes \mathcal{I}_x$ and induces an isomorphism $\mathcal{I}_x/\mathcal{I}_x^2 \simeq \mathcal{H}^{1,0}(M)/\mathcal{H}^{1,0}(M) \otimes \mathcal{I}_x$.

Definition 6.1.4. A holomorphic line bundle L on a compact complex manifold M is called *very ample*, if the map Φ_L is an embedding, i.e. L separates points and tangent directions. L is called *ample* if some tensor power $L^k = L^{\otimes k}$, $k \in \mathbb{N}$, is very ample.

The following is the immediate consequence of the definition:

Proposition 6.1.5. *Let M be a compact complex manifold and suppose that there exists an ample line bundle on M. Then M is projective.* □

Example 6.1.6. Returning to the Veronese embedding, it is clear that homogeneous polynomials of degree $d > 0$ separate points and tangent directions, so this really is an embedding.

Example 6.1.7 (Elliptic curves). Let $C = \mathbb{C}/\Lambda$ be an elliptic curve, where $\Lambda = \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}$, $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. We consider line bundles $\mathcal{O}(kp) = [kp]$ corresponding to divisors of the form kp, where p is a point on C and $k \in \mathbb{N}$. We have exact sequences:

$$0 \longrightarrow \mathcal{O}((k-1)p) \longrightarrow \mathcal{O}(kp) \longrightarrow \mathcal{O}(kp)/\mathcal{O}((k-1)p) \simeq \mathbb{C} \longrightarrow 0, \tag{6.1.4}$$

where the first map multiplies a local section of $\mathcal{O}((k-1)p)$ on U by $z-p$ if $p \in U$ and by 1 if $p \notin U$. Since K_C is trivial (C is a torus), the Kodaira-Serre duality tells us that $H^1(C, \mathcal{O}(kp)) \simeq H^0(C, \mathcal{O}(-kp))^*$ and hence $\dim H^1(C, \mathcal{O}(kp))$ is 1 if $k = 0$ and 0 if $k \geq 1$. The long exact cohomology sequence of (6.1.4) shows now inductively that $\dim H^0(C, \mathcal{O}(kp)) = k$.

We consider the maps Φ_L corresponding to $L = \mathcal{O}(kp)$, $k \in \mathbb{N}$. Observe that the ideal sheaves occuring in (6.1.1)-(6.1.3) are now line bundles: $\mathcal{O}(kp) \otimes \mathcal{I}_x \simeq \mathcal{O}(kp - x)$, $\mathcal{O}(kp) \otimes \mathcal{I}_{x,y} \simeq \mathcal{O}(kp - x - y)$, $\mathcal{O}(kp) \otimes \mathcal{I}_x^2 \simeq \mathcal{O}(kp - 2x)$. The long exact sequence of (6.1.4) with $k = 1$ shows that $\mathcal{O}(p)$ is not generated by sections. $\mathcal{O}(2p)$ is generated by sections ($H^1(C, \mathcal{O}(2p - x)) = 0$ due to the Serre duality), but it separates neither points nor tangent directions (non-separation of points can be seen from topology: there is no smooth injective map from a torus to $\mathbb{P}^1 \simeq S^2$; for tangent directions set $x = p$ in (6.1.3) - then $L \otimes \mathcal{I}_x^2 \simeq \mathcal{O}$ and the long exact sequence of (6.1.3) shows that d_x is not surjective). For $k \geq 3$, however, $\mathcal{O}(kp)$ is very ample. Indeed, the Serre duality implies then that $H^1(C, L \otimes \mathcal{I}_{x,y}) = 0$ and $H^1(C, L \otimes \mathcal{I}_x^2) = 0$, so the corresponding maps on global sections are surjective.

Thus, for $L = \mathcal{O}(3p)$, the map $\Phi_L : C \to \mathbb{CP}^2$ is an embedding. In order to identify it, we need to describe global sections of $\mathcal{O}(3p)$. Recall from §3.5 that $\mathcal{O}(kp)$ has a tautological section s_0 with zero of order k at p. If s is any other global section of $\mathcal{O}(kp)$ then s/s_0 is a meromorphic section of $\mathcal{O}(kp) \otimes \mathcal{O}(kp)^* = \mathcal{O}(kp - kp) = \mathcal{O}$, i.e. a meromorphic function on C. Moreover the only singularity of s/s_0 is a pole of order at most k at p. Conversely, if f is a meromorphic function on C with the only singularity a pole of order at most k at p (usual notation is $(f) \geq -kp$), then fs_0 is a holomorphic section of $\mathcal{O}(kp)$. We have thus an isomorphism of vector spaces:

$$H^0(C, \mathcal{O}(kp)) \simeq \{f \in H^0(C, \mathcal{M});\ (f) \geq -kp\}. \tag{6.1.5}$$

The section s_0 corresponds to the constant function $f = 1$. Taking $k = 1$ in the above correspondence shows that there is no meromorphic function on C with exactly one simple pole at p. Setting $k = 2$ shows that there is a unique (up to rescaling) meromorphic function with pole of order 2 at p. This function, known as the Weierstraß $\wp$-function, can be written explicitly. For $z \in \mathbb{C} \backslash \Lambda$ set

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \backslash \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

This is a Λ-periodic meromorphic function on $\mathbb{C}$ (I shall leave the convergence of the series as an exercise) with doubles poles at points of Λ, and hence it descends to a meromorphic function on C with a double pole at the point p corresponding to $0 \in \mathbb{C}$. The derivative of $\wp$

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^3}$$

has a pole of order 3 at p, and therefore corresponds to a section of $\mathcal{O}(3p)$. The functions 1 and $\wp$ also correspond to global sections of $\mathcal{O}(3p)$ under (6.1.5), and since they all have poles of different order, they are linearly independent. Therefore:

$$H^0(C, \mathcal{O}(3p)) \simeq \langle 1, \wp, \wp' \rangle.$$

The corresponding map $\Phi_L : C \to \mathbb{CP}^2$ is then

$$z \mapsto [1, \wp(z), \wp'(z)]. \tag{6.1.6}$$

We can identify its image as follows. Observe that the following seven functions $1, \wp, \wp', \wp^2, \wp\wp', \wp^3, (\wp')^2$ all have a pole of order at most 6 at p, and hence they correspond to sections of $\mathcal{O}(6p)$. However $\dim H^0(C, \mathcal{O}(6p)) = 6$, and therefore these functions are linearly dependent. Comparing the coefficients of z^{-6} shows that the relation among them is of the form

$$(\wp')^2 = 4\wp^3 + a_1\wp\wp' + a_2\wp^2 + a_3\wp' + a_4\wp + a_5, \tag{6.1.7}$$

for some constants $a_1, \ldots, a_5$.[2] Therefore the image of the map (6.1.6) is defined by a cubic equation. Conversely, it follows from Ex. 2 in Homework 5 that a smooth projective plane curve $C \subset \mathbb{CP}^2$, defined by a cubic equation, satisfies $\dim H^1(C, \mathcal{O}) = 1$, i.e. it has genus 1, and is therefore an elliptic curve (embedded in $\mathbb{CP}^2$ by Φ_L with $L \simeq \mathcal{O}(p_1 + p_2 + p_3)$ for some $p_1, p_2, p_3 \in C$).

Let me finish this long, but hopefully instructive example by considering the embedding $\Phi_L : C \to \mathbb{CP}^3$ corresponding to $L = \mathcal{O}(4p)$. Arguments completely similar to the ones above show that

$$H^0(C, \mathcal{O}(4p)) \simeq \langle 1, \wp, \wp', \wp^2 \rangle,$$

and consequently the map Φ_L is

$$z \mapsto [1, \wp(z), \wp'(z), \wp^2(z)].$$

Set $X = \wp(z)$, $Y = \wp'(z)$, $Z = \wp^2(z)$, and observe that $Z = X^2$ and that the relation (6.1.7) can be now written as:

$$Y^2 = 4XZ + a_1XY + a_2Z + a_3Y + a_4X + a_5.$$

In other words, $\Phi_L(C)$ is cut out by two quadratic equations. Conversely, one can show that a smooth intersection of two (distinct) quadrics in $\mathbb{CP}^3$ is an elliptic curve.

6.2 Kodaira embedding theorem

Theorem 6.2.1 (Kodaira)**.** *A holomorphic line bundle on a compact complex manifold is ample if and only if it is positive.*

Proof. One direction is easy: if L is ample, then there exists a $k \in \mathbb{N}$ such that L^k is the pullback of the hyperplane line bundle on the projective space. Hence $kc_1(L)$ is the pullback of $c_1(\mathcal{O}(1))$, therefore positive.

For the other direction, we need the following result from analysis:

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[2] A more precise analysis will show that $a_1 = a_2 = a_3 = 0$.

Theorem 6.2.2 (Hartog's theorem)**.** *Let $\Delta(r)$ and $\Delta(r')$ be two closed polydisks in $\mathbb{C}^n$ with $r > r'$ and $n \geq 2$. Any holomorphic function f defined on a neighbourhood of $\Delta(r)\backslash\Delta(r')$ extends to a holomorphic function on $\Delta(r)$.*

Proof. In order to keep the notation simple, we assume that $n = 2$ (the general case is then straightforward). Let z_1, z_2 be complex coordinates on $\mathbb{C}^2$ and observe that each slice $z_1 = \text{const}$ of $\Delta(r)\backslash\Delta(r')$ is either the annulus $r' < |z_2| \leq r$, or the disk $|z_2| \leq r$. Use the Cauchy formula and define

$$F(z_1, z_2) = \frac{1}{2\pi i} \int\limits_{|w_2|=r} \frac{f(z_1, w_2)}{w_2 - z_2} dw_2.$$

F is defined on $\Delta(r)$ and clearly holomorphic. Moreover, in the open connected subset $|z_1| > r'$ of $\Delta(r)\backslash\Delta(r')$

$$F(z_1, z_2) = \operatorname*{res}_{w_2=z_2} \frac{f(z_1, w_2)}{w_2 - z_2} = f(z_1, z_2),$$

and so $F = f$ on this open subset; hence $F = f$ on $\Delta(r)\backslash\Delta(r')$. □

We need to show that for any positive line bundle L there exists $k \in \mathbb{N}$ such that L^k separates points and tangent directions. I hope that by now you noticed how useful divisors are, so the technique used here, and in many similar situations, is to replace a point, which has a high codimension, with its blow-up, which is a divisor. Let x and y be two distinct points of M and blow up M at both of them. Denote the result by $\widetilde{M}$ and let $\pi : \widetilde{M} \to M$ be the natural projection. Put $\widetilde{L} = \pi^* L$ and consider the pullback map on sections

$$\pi^* : H^0(M, L^k) \to H^0(\widetilde{M}, \widetilde{L}^k).$$

Any section of $\widetilde{L}^k$ defines a section of L^k on $M\backslash\{x, y\}$. For $n \geq 2$ it extends to a section on all of M owing to Hartog's theorem, while for $n = 1$ $\widetilde{M} = M$ and π^* is identity. Therefore the map π^* is an isomorphism on global sections.

Furthermore, since $\widetilde{L}^k = \pi^* L^k$, $\widetilde{L}^k$ is trivial along the exceptional divisors E_x and E_y, i.e.

$$\widetilde{L}^k\big|_{E_x} = E_x \times L^k\big|_x, \qquad \widetilde{L}^k\big|_{E_y} = E_y \times L^k\big|_y.$$

Let $E = E_x \cup E_y$ and denote by r_E (resp. by r_{xy}) the restriction of sections to E (resp. to $\{x, y\}$). The above considerations imply that we have a commutative diagram:

$$\begin{tikzcd}
H^0(\widetilde{M}, \widetilde{L}^k) \arrow[r, "r_E"] & H^0(E, \widetilde{L}^k) \arrow[d, equal] \\
H^0(M, L^k) \arrow[u, "\pi^*"] \arrow[r, "r_{xy}"] & L^k\big|_x \oplus L^k\big|_y.
\end{tikzcd}$$

Therefore, in order to prove that L^k separates points, it suffices to show that r_E is surjective. The kernel of the sheaf map $r_E : \widetilde{L}^k \to \widetilde{L}^k|_E$ is the sheaf of sections which vanish on E, i.e. the sheaf of local sections of $\widetilde{L}^k \otimes [-E]$ (it is at this point that replacing points by divisors pays off). Let us abbreviate the line bundle $\widetilde{L}^k \otimes [-E]$ to $\widetilde{L}^k(-E)$. Thus we have a short exact sequence

$$0 \longrightarrow \widetilde{L}^k(-E) \longrightarrow \widetilde{L}^k \xrightarrow{r_E} \widetilde{L}^k\big|_E \longrightarrow 0,$$

and the surjectivity of r_E on global sections is equivalent to $H^1(\widetilde{M}, \widetilde{L}^k(-E)) = 0$. Since the sheaf $\mathcal{H}^{n,0}$ of holomorphic forms of highest degree on $\widetilde{M}$ is isomorphic to $K_{\widetilde{M}}$, we can write

$$\widetilde{L}^k(-E) \simeq \mathcal{H}^{n,0}\left(\widetilde{L}^k(-E) \otimes K^*_{\widetilde{M}}\right).$$

The Kodaira-Akizuki-Nakano vanishing theorem (Theorem 4.3.3) implies that $H^1(\widetilde{M}, \widetilde{L}^k(-E)) = 0$, provided that $\widetilde{L}^k(-E) \otimes K^*_{\widetilde{M}}$ is a positive line bundle on $\widetilde{M}$.

We have seen in the previous chapter (Lemma 5.5.2) that $c_1(\widetilde{M}) = \pi^* c_1(M) + (n-1)c_1([-E])$. Therefore we need to prove the positivity of

$$c_1\left(\widetilde{L}^k\right) + \pi^* c_1(M) + n c_1([-E]) = k\pi^* c_1(L) + \pi^* c_1(M) + n c_1([-E]). \quad (6.2.1)$$

We also know (Lemma 5.5.2 again) that $[-E_x]\big|_{E_x} \simeq H$, where H is the hyperplane line bundle on $E_x \simeq \mathbb{CP}^{n-1}$, and similarly for E_y. Therefore $[-E]\big|_E$ has a hermitian metric which has a positive (Ricci) curvature. This metric can be extended to a neighbourhood of E and the curvature will stay positive on some smaller neighbourhood. On the other hand, the bundle $[E]$ has a tautological section s vanishing exactly on E. In other words s trivialises $[E]$ over $M \backslash E$ and we can define a flat hermitian metric on $[E]\big|_{\widetilde{M} \backslash E}$, and hence on $[-E]\big|_{\widetilde{M} \backslash E}$, by setting $|s|^2 = 1$. We can now glue these two metrics using a bump function, and obtain a hermitian metric on $[-E]$, the Ricci form ρ of which is positive on a neighbourhood U_1 of E and identically zero outside a neighbourhood $U_2 \supset U_1$. By assumption L has a hermitian metric, the curvature ϕ of which is positive. Let $[i\psi]$ represent $c_1(M)$. For sufficiently large k_1, $k_1\phi + \psi$ is positive. The pullback of $k_1\phi + \psi$ to $\widetilde{M}$ is positive outside E, while at the points of E $\pi^*(k_1\phi + \psi)(v, \bar{v}) = 0$ if v is tangent to E and is positive if v is normal to E. It follows that for a sufficiently large k_2, the form

$$\pi^*(k_1\phi + \psi) + (k_2\pi^*\phi + n\rho)$$

is positive, which proves the positivity of (6.2.1) for $k = k_1 + k_2$.

We have shown that for every pair of distinct points x, y, there exists $k \in \mathbb{N}$ such that L^k separates x and y. We still need to show that k can be chosen independently of x and y. But, clearly, if L^k separates x and y, then it separates nearby points, so this claim follows from the compactness of M.

Separation of tangent vectors is proved similarly. Let $x \in M$ and let $\pi : \widetilde{M} \to M$ be now the blow-up of M at x, with $E = \pi^{-1}(x)$. Again the pullback map

$$\pi^* : H^0\left(M, L^k\right) \longrightarrow H^0\left(\widetilde{M}, \widetilde{L}^k\right)$$

is an isomorphism ($\widetilde{L} = \pi^* L$). Furthermore, if $\sigma \in H^0(M, L^k)$, then $\sigma(x) = 0$ is equivalent to $\pi^*\sigma$ vanishing on E. Therefore π^* restricts to an isomorphism

$$\pi^* : H^0\left(M, L^k \otimes \mathcal{I}_x\right) \longrightarrow H^0\left(\widetilde{M}, \widetilde{L}^k(-E)\right).$$

The bundle $[-E]\big|_E$ is identified with the conormal bundle N_E^* of E, and hence $H^0\left(E, [-E]\big|_E\right) \simeq T_x^* M$. We obtain a commutative diagram

$$\begin{tikzcd}
H^0(\widetilde{M}, \widetilde{L}^k[-E]) \arrow[r, "r_E"] & H^0(E, \widetilde{L}^k[-E]) \arrow[d, equal] \\
H^0(M, L^k \otimes \mathcal{I}_x) \arrow[u, "\pi^*"] \arrow[r, "d_x"] & L^k\big|_x \otimes T_x^* M.
\end{tikzcd}$$

We must show that r_E is surjective for large k. We have an exact sequence of sheaves on $\widetilde{M}$:

$$0 \longrightarrow \widetilde{L}^k[-2E] \longrightarrow \widetilde{L}^k[-E] \xrightarrow{r_E} \widetilde{L}^k[-E]\Big|_E \longrightarrow 0,$$

and the long exact sequence on cohomology implies that the surjectivity of r_E is equivalent to $H^1\big(\widetilde{M}, \widetilde{L}^k[-2E]\big) = 0$.

As before $\widetilde{L}^k[-2E] \simeq \mathcal{H}^{n,0}_{\widetilde{M}}\left(\widetilde{L}^k[-2E] \otimes K^*_{\widetilde{M}}\right)$ as sheaves. The same argument as for x, y shows that $\widetilde{L}^k[-2E] \otimes K^*_{\widetilde{M}}$ is positive for large k, and again the Kodaira-Akizuki-Nakano vanishing theorem implies that

$$H^1(\widetilde{M}, \widetilde{L}^k[-2E]) = H^1\left(\widetilde{M}, \Omega^n_{\widetilde{M}}\left(\widetilde{L}^k[-2E] \otimes K^*_{\widetilde{M}}\right)\right) = 0.$$

Again k can be chosen independently of x. ☐

Projectivity of complex manifolds

Theorem 6.2.1 can be reformulated as follows:

Theorem 6.2.3 (Kodaira embedding theorem)**.** *A compact complex manifold is projective if and only if it has a closed positive* $(1,1)$*-form* ω *such that* $[\omega]$ *is rational.*

Proof. If M is projective, then the Chern class of the hyperplane line bundle restricted to M is positive and integer. Conversely, suppose that we have a form ω as in the statement. Then $[k\omega] \in H^2(M, \mathbb{Z})$ for some $k \in \mathbb{N}$. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

yields

$$\cdots \longrightarrow H^1(M,\mathcal{O}^*) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \xrightarrow{i_*} H^2(M,\mathcal{O}) \longrightarrow \ldots$$

Since $H^2(M,\mathcal{O}) \simeq H^{0,2}_{\bar\partial}(M)$, i_* maps any $(1,1)$-class to zero. Therefore $i_*([k\omega]) = 0$ and there is a line bundle $L \in H^1(M,\mathcal{O}^*)$ with $c_1(L) = [k\omega]$. L is positive, hence M is projective, owing to Theorem 6.2.1 and Proposition 6.1.5. ☐

On a Kähler manifold we can consider the subset $\mathcal{K}$ of $H^{1,1}_{\bar\partial}(M) \cap H^2(M,\mathbb{R})$ consisting of positive forms. This is called the *Kähler cone* of M and is an open cone (i.e. a convex subset closed under multiplication by positive scalars). The above theorem says that M is projective if and only if $\mathcal{K} \cap H^2(M,\mathbb{Q}) \neq \emptyset$ (or $\mathcal{K} \cap H^2(M,\mathbb{Z}) \neq \emptyset$).

A simple sufficient condition is given by:

Corollary 6.2.4. *A compact Kähler manifold M with $H^{0,2}_{\bar\partial}(M) = 0$ is projective.*

Proof. In this case $H^{1,1}_{\bar\partial}(M) = H^2(M,\mathbb{C}) = H^2(M,\mathbb{Z}) \otimes \mathbb{C}$. An open cone (such as $\mathcal{K}$) must intersect the integer lattice $H^2(M,\mathbb{Z})$. ☐

We finish the section by showing that several standard constructions preserve projectivity.

Corollary 6.2.5. *If M_1 and M_2 are projective, then so is $M_1 \times M_2$.*

Proof. If ω_1, ω_2 are rational closed positive $(1,1)$-forms on M_1, M_2, respectively, and $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, are the projections, then $\pi_1^*\omega_1 + \pi_2^*\omega_2$ is again a closed rational positive $(1,1)$-form. ☐

Example 6.2.6 (Segre map). This is an embedding

$$\mathbb{CP}^n \times \mathbb{CP}^m \hookrightarrow \mathbb{CP}^N$$

given by Φ_L (§6.1) for the very ample line bundle $L = \pi_1^* H_{\mathbb{CP}^n} \otimes \pi_2^* H_{\mathbb{CP}^m}$ on $\mathbb{CP}^n \times \mathbb{CP}^m$. For example, the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ into $\mathbb{CP}^3$ is

$$\big([z_0, z_1], [w_0, w_1]\big) \longmapsto [x_0, x_1, x_2, x_3] = [z_0w_0, z_0w_1, z_1w_0, z_1w_1].$$

Its image is the quadratic surface $x_0x_3 = x_1x_2$ in $\mathbb{CP}^3$.

Corollary 6.2.7. *If M is projective, then the blow-up $\widetilde{M}$ of M at a point is projective.*

Proof. The proof of the theorem 6.2.1 shows that if L is positive, then $\widetilde{L}^k[-E]$ is positive for large k. ☐

Corollary 6.2.8. *If $\widetilde{M} \to M$ is a finite covering of compact complex manifolds, then $\widetilde{M}$ is projective if and only if M is.*

Proof. The induced map on cohomology $H^2(M,\mathbb{C}) \to H^2(\widetilde{M},\mathbb{C})$ is just the division by the number of sheets of the covering. Moreover it preserves positivity. Hence there is a positive closed $(1,1)$-form in $H^2(M,\mathbb{Q})$ if and only if there is one in $H^2(\widetilde{M},\mathbb{Q})$. ☐

6.3 Further properties of projective manifolds

There are several interesting results which are valid only for projective manifolds. As you may guess, the reason is the existence of an ample (i.e. positive) line bundle on such manifolds.

Line bundles and divisors II

We are going to prove the property already mentioned in §3.5, namely that every line bundle on a projective manifold is associated to a divisor. We need first a result, which is a version of Sard's theorem in the special case of linear systems. A *linear system* on a complex manifold is a subspace V of $H^0(M, L)$ for some line bundle L. The *base locus* B of a linear system is the set of all points $x \in M$ such that $s(x) = 0$ for all $s \in V$.

Lemma 6.3.1 (Bertini's theorem)**.** *Let V be a linear system on a complex manifold M, with base locus B. For a generic $s \in V$, $s^{-1}(0)\backslash B$ is smooth.*

Proof. Fix a basis $s_1, \ldots, s_k$ of V and consider the map $\phi : \big(M\backslash B\big) \times \mathbb{C}^k \to \mathbb{C}$ given by

$$\phi(x, \alpha) = \phi\big(x, (\alpha_1, \ldots, \alpha_k)\big) = \sum_{i=1}^{k} \alpha_i s_i(x).$$

Since $x \notin B$, there is an i such that $\frac{\partial \phi}{\partial \alpha_i}\big|_{(x,\alpha)} \neq 0$, and hence $d\phi$ is surjective at every point (x, α). Consequently $\phi^{-1}(0)$ is smooth. Now consider the projection $\pi : \phi^{-1}(0) \to \mathbb{C}^k$. Sard's theorem implies that for a generic choice of $\alpha = (\alpha_1, \ldots, \alpha_k)$, the set $\pi^{-1}(\alpha) = \big(\sum_{i=1}^{k} \alpha_i s_i\big)^{-1}(0)\backslash B$ is smooth. □

Proposition 6.3.2. *Let M be a projective manifold. Then the natural map $\mathrm{Div}(M) \to \mathrm{Pic}(M)$ is surjective.*

Proof. Let M be an n-dimensional compact complex manifold embedded in some $\mathbb{CP}^N$. We have to show that every line bundle L on M has a meromorphic section. First of all, I claim that $H^1(M, L(k)) = 0$ for sufficiently large $k \in \mathbb{N}$, where $L(k)$ denotes the tensor product of L with the restriction of $\mathcal{O}_{\mathbb{CP}^N}(k)$ to M. This is the same trick as in the proof of the Kodaira theorem: we view $L(k)$ as the sheaf $\mathcal{H}^{n,0}(L(k) \otimes K_M^*)$ of holomorphic n-forms with values in $L(k) \otimes K_M^*$. Since $\mathcal{O}(1)$ is positive and M is compact, $L(k) \otimes K_M^*$ will be positive for large k, and the claim follows from the Kodaira-Akizuki-Nakano vanishing theorem.

Now I claim that for sufficiently large k, the bundle $L(k)$ has a global holomorphic section. We prove this by induction on $\dim M$ (i.e. we prove that for any compact submanifold M of $\mathbb{CP}^N$ and any line bundle L on M, $H^0(M, L(k)) \neq 0$ for large enough k). The claim is trivial if $\dim M = 0$. Suppose that the statement holds for all $(n-1)$-dimensional projective submanifolds of $\mathbb{CP}^N$. According to the above lemma, we can find a section s of $\mathcal{O}_{\mathbb{CP}^N}(1)$ such that $D = s^{-1}(0) \cap M$ is smooth (since $\mathcal{O}(1)$ is base-free). Now consider the exact sequence:

$$0 \longrightarrow L(k-1) \xrightarrow{\cdot s} L(k) \longrightarrow L(k)|_D \longrightarrow 0.$$

Let k be large enough so that $H^1(M, L(k-1)) = 0$ and $H^0(D, L(k)) \neq 0$ (such a k exists by inductive assumption). The long exact sequence implies that $H^0(M, L(k)) \neq 0$, proving the claim. We finish the proof by observing that if t is any holomorphic section of $L(k)$ and p is any homogeneous polynomial of degree k (i.e. a section of $\mathcal{O}(k)$), then t/p is a meromorphic section of L. $\square$

Lefschetz hyperplane section theorem[3]

Theorem 6.3.3. *Let M be a compact Kähler manifold with $\dim_{\mathbb{C}} M = n$ and $V \subset M$ a smooth compact complex hypersurface such that the line bundle $[V]$ is positive. Then the map*

$$H^q_{\mathrm{dR}}(M) \to H^q_{\mathrm{dR}}(V),$$

induced by the inclusion $V \hookrightarrow M$, is an isomorphism for $q \leq n-2$ and injective for $q = n-1$.

Proof. Both M and V are compact Kähler, hence their de Rham cohomology admits the Hodge decomposition. Thanks to the Dolbeault theorem we have to prove that

$$H^q(M, \mathcal{H}^{p,0}_M) \to H^q(V, \mathcal{H}^{p,0}_V)$$

is an isomorphism if $p+q \leq n-2$ and injective if $p+q = n-1$. We have two short exact sequences of sheaves

$$0 \longrightarrow \mathcal{H}^{p,0}_M(-V) \longrightarrow \mathcal{H}^{p,0}_M \xrightarrow{r} \mathcal{H}^{p,0}_M\big|_V \longrightarrow 0, \tag{6.3.1}$$

where $\mathcal{H}^{p,0}_M(-V)$ is the sheaf of forms vanishing on V, and the conormal sequence (recall Ex. 2 in Homework 7)

$$0 \longrightarrow [-V]\big|_V \longrightarrow \mathcal{H}^{1,0}_M\big|_V \xrightarrow{i} \mathcal{H}^{1,0}_V \longrightarrow 0.$$

Taking the p-th exterior power of this last sequence yields:[4]

$$0 \longrightarrow \mathcal{H}^{p-1}_V(-V) \longrightarrow \mathcal{H}^{p,0}_M\big|_V \xrightarrow{i} \mathcal{H}^{p,0}_V \longrightarrow 0. \tag{6.3.2}$$

By assumption $[-V]$ is negative on M, and hence $[-V]\big|_V$ is negative. The Kodaira-Akizuki-Nakano vanishing theorem (cf. Remark 4.3.7) implies that

$$H^q\big(M, \mathcal{H}^{p,0}_M(-V)\big) = 0 \text{ if } p+q < n \quad \text{and} \quad H^q\big(V, \mathcal{H}^{p-1,0}_V(-V)\big) = 0 \text{ if } p+q < n.$$

Now taking the long exact sequences of (6.3.1) and (6.3.2) shows that the composition

$$H^q\big(M, \mathcal{H}^{p,0}_M\big) \xrightarrow{r^*} H^q\big(V, \mathcal{H}^{p,0}_M\big|_V\big) \xrightarrow{i^*} H^q\big(V, \mathcal{H}^{p,0}_V\big)$$

is an isomorphism if $p+q < n-1$ and injective if $p+q = n-1$. $\square$

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[3] Also known as the *weak Lefschetz theorem.*

[4] I shall leave the linear algebra argument as an exercise.

Example 6.3.4. Taking $M = \mathbb{CP}^n$ and V a hypersurface defined by a homogeneous polynomial of degree d (so that $[V] = \mathcal{O}(d)$) shows that the cohomology of V is the same as that of $\mathbb{CP}^n$ up to dimension $n-2$. Now the Poincaré duality implies that for $q \geq n$ $H^q_{\mathrm{dR}}(V) \simeq H^{q-2}_{\mathrm{dR}}(\mathbb{CP}^n)$. The only cohomology group which is not completely determined is the middle one $H^{n-1}_{\mathrm{dR}}(V)$. This can indeed be much larger than $H^{n-1}_{\mathrm{dR}}(\mathbb{CP}^n)$ - recall exercise 2 from Homework 5, where you showed that such a V in $\mathbb{CP}^2$ is a Riemann surface of genus $\binom{d-1}{2}$, and hence $\dim H^1_{\mathrm{dR}}(V) = (d-1)(d-2)$.

Example 6.3.5. A projective submanifold X of $\mathbb{CP}^n$ with $\dim_{\mathbb{C}} X = k$ is called a *complete intersection* if it is defined by $n-k$ homogeneous polynomials. Applying the Lefschetz theorem repeatedly shows that $H^q_{\mathrm{dR}}(X) \simeq H^q_{\mathrm{dR}}(\mathbb{CP}^n)$ if $q < k$. This allows us to immediately tell that many projective manifolds cannot be complete intersections. This is the case, for example, for any projective torus of dimension greater than 1 (since $H^1_{\mathrm{dR}}(\mathbb{CP}^n) = 0$).

Chow theorem

In its original formulation, Serre's famous GAGA theorem (see p. 32) asserts the equivalence of categories of coherent algebraic sheaves on a projective variety and the category of coherent analytic sheaves on the corresponding analytic space. Several years before (in 1949) W.-L. Chow proved that projective analytic varieties are algebraic. I shall now give a proof of this.

First of all, let us define subvarieties.

Definition 6.3.6. Let M be a complex manifold. A subset $X \subset M$ is called an *analytic subvariety* of M if every point $x \in X$ has a neighbourhood U in M such that $X \cap U$ is the common zero set of a finite number of holomorphic functions defined on U.

Definition 6.3.7. A subset X of a projective space $\mathbb{CP}^n$ is called an *algebraic subvariety* if it is the common zero set of a number of homogeneous polynomials.

In both cases a subvariety is said to be *irreducible* if it is not the union of two other subvarieties.

Theorem 6.3.8 (Chow theorem)**.** *Every compact analytic subvariety of* $\mathbb{CP}^n$ *is algebraic.*

Proof. We have essentially proved this already in the case of an hypersurface. If V is a compact analytic hypersurface in $\mathbb{CP}^n$, then the line bundle $[V]$ has a holomorphic section vanishing on V. However, any line bundle on $\mathbb{CP}^n$ is a power of the hyperplane line bundle (Remark 4.3.9), and consequently, any section of $[V]$ is a homogeneous polynomial.

For subvarieties of higher codimension we are going to use the technique of projections, which is of interest on its own. If L is an $(n-k-1)$-dimensional projective subspace of $\mathbb{P}^n$ (i.e. L is the projectivisation of an $(n-k)$-dimensional linear subspace of $\mathbb{C}^{n+1}$), then we can project $\mathbb{CP}^n \backslash L$ onto any complementary

k-dimensional projective subspace $\Lambda \simeq \mathbb{CP}^k$ by sending a point $q \in \mathbb{CP}^n \backslash L$ to the intersection of $\langle q, L\rangle$ with Λ. If we choose linear coordinates so that

$$L = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n;\ z_0 = \cdots = z_k = 0\},$$

$$\Lambda = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n;\ z_{k+1} = \cdots = z_n = 0\},$$

then this projection is simply

$$\pi : [z_0, \ldots, z_n] \longmapsto [z_0, \ldots, z_k].$$

Let now X be a k-dimensional compact analytic subvariety of $\mathbb{CP}^n$ and $p \notin X$. Choose an $(n - k - 2)$-dimensional projective subspace L disjoint from p and such that the $(n-k-1)$-dimensional subspace $\langle p, L\rangle$ is disjoint from X. Project $\mathbb{CP}^n \backslash L$ onto a complementary $\mathbb{CP}^{k+1}$. If we can show that the image of X is still an analytic subvariety, then owing to the argument at the beginning of the proof, $\pi(X)$ is the zero locus of a homogeneous polynomial $f(z_0, \ldots, z_{k+1})$. This polynomial, viewed as a polynomial in $n + 1$ variables, vanishes on X but not at p, since $\pi(p) \notin \pi(X)$. Therefore, for every $p \in \mathbb{CP}^n$, there is a homogeneous polynomial vanishing on V, but not at p, and Chow's theorem follows from Hilbert's basis theorem.

Thus the proof of Chow's theorem is reduced to showing that a projection of an analytic subvariety is an analytic subvariety. A projection from an $(n - k - 2)$-dimensional subspace L can be replaced by repeated projections from a point, so we only need to show that if $q \notin X$, then the image of a compact k-dimensional analytic subvariety under the projection $\mathbb{CP}^n \backslash \{q\}$ to $\mathbb{CP}^{n-1}$ is an analytic subvariety.

Since the property of being analytic is local, it is sufficient to show that if Y is an analytic subvariety of a neighbourhood of 0 in $\mathbb{C}^n$, and the line $z_1 = \cdots = z_{n-1} = 0$ is not contained in Y, then the image of a neighbourhood of 0 in Y under the projection

$$\pi : (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_{n-1})$$

is an analytic subvariety of a neighbourhood of 0 in $\mathbb{C}^{n-1}$.

Let Y be given, in a neighbourhood of 0, as the common zero locus of finitely many holomorphic functions $f_1, \ldots, f_r$. We may assume (replacing the f_i with their linear combinations, if necessary) that no f_i is identically zero along the z_n-axis. The Weierstraß preparation theorem[5] implies that we can replace the f_i with functions which are polynomials in z_n, i.e. each such function is of the form

$$h(w, z_n) = \sum_{j=1}^{d} a_j(w) z_n^j, \tag{6.3.3}$$

where each coefficient is a holomorphic function of $w = (z_1, \ldots, z_{n-1})$ in a neighbourhood of $0 \in \mathbb{C}^{n-1}$. For any polynomial in one variable, its coefficients

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[5] See Griffiths and Harris, pp. 7–8.

are given by the elementary symmetric polynomials in its roots $t_1, \dots, t_d$. On the other hand any symmetric polynomial in $t_1, \dots, t_d$ can be expressed as a polynomial in elementary symmetric polynomials. Therefore, if $t_1(w), \dots, t_d(w)$ denote the roots of $h(w, \cdot)$, where h is of the form (6.3.3), then

$$\bar{h}(w) = \prod_{i=1}^{d} h(t_i(w))$$

is a well-defined holomorphic function in a neighbourhood of $0 \in \mathbb{C}^{n-1}$. It is easy to verify that $\pi(Y)$ is the common zero locus of functions $\overline{f_1}, \dots, \overline{f_r}$.

□

Remark 6.3.9. The proof given here also shows that if $f : M \to M'$ is a holomorphic submersion between complex manifolds and X is an analytic subvariety of M such that $f|_X$ is finite-to-one, then $f(X)$ is an analytic subvariety of M'. This is a special case of Remmert's *proper mapping theorem* which asserts that $f(X)$ is an analytic subvariety for any holomorphic map $f : M \to M'$ such that $f|_X$ is proper. The proof of this is hard; see Griffiths & Harris, pp. 395ff., for a proof under an additional assumption, or H. Grauert and R. Remmert *"Coherent analytic sheaves"* (Springer 1984) for a proof in full generality.

THE END