Curve Shortening on Sasaki Manifolds and the Weinstein Conjecture

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Abstract

We propose a new geometric evolution equation for closed curves on contact manifolds. Our flow is a parabolic evolution equation similar to the well known curve shortening flow (mean curvature flow) and is the negative gradient flow of the Reeb energy of closed curves. We analyze the longtime and singular behavior of the flow and describe relations to the Weinstein conjecture on the existence of closed Reeb orbits.

Keywords: curve shortening, mean curvature flow, Reeb orbit, Sasaki manifold, Weinstein conjecture.

1 Introduction

The Weinstein conjecture [11] asserts that each regular compact contact type level set of a Hamiltonian on a symplectic manifold carries at least one periodic orbit of the corresponding Hamiltonian flow. Viterbo [10] proved this conjecture in its original form for Hamiltonians on linear symplectic spaces. In general, the conjecture is still widely open but there are many partial results, often under additional assumptions.

In general one can rephrase the conjecture in terms of contact geometry. It then asserts that on any compact contact manifold (M^{2n+1}, λ) there exists at least one closed Reeb orbit, i.e. a periodic integral curve of the associated Reeb vector field X. The Weinstein conjecture has been very influential in symplectic geometry and topology and led to many new insights and theories, some of them extremely involved. For example, with the theory of pseudoholomorphic curves, Hofer [6] proved the existence of periodic orbits on S^3 and on closed three-dimensional contact manifolds M with $\pi_2(M) = 0$ and in addition for the Reeb flow of an overtwisted contact form. Taubes [8], [9] recently made a break-through and proved the conjecture for any closed three-dimensional contact manifold. In his proof he uses a variant of Seiberg-Witten Floer homology. More recently, Albers and Hofer [1] proved the existence of closed Reeb orbits in higher dimensions, if there exists a so called *Plastikstufe*. More references on the Weinstein conjecture can be found in the nice survey article by V. Ginzburg [4].

The aim of this paper is to introduce a completely different approach via some geometric evolution equations that are similar to the well known curve shortening flow (cf. [3], [5]) in Riemannian geometry. Suppose λ is a contact form on an odd-dimensional compact manifold M^{2n+1}, i.e. $\lambda \wedge d\lambda^n$ is a volume form. Associated to such a contact manifold (M, λ) is the Reeb vector field X, uniquely determined by the conditions

$$i_X \lambda = 1\,, \quad i_X d\lambda = 0\,.$$

In addition $\xi = \ker \lambda$ is the underlying contact distribution of M and

$$\pi : TM \to \xi\,, \quad W \mapsto W - \lambda(W)X$$

is the natural projection of TM onto ξ.

Suppose now that $\gamma : S^1 \to M$ is a regular closed smooth curve in M. Then we have:

$$\gamma \text{ is a closed Reeb orbit } \Leftrightarrow i_{\gamma'(\phi)} d\lambda(\gamma(\phi)) = 0\,, \forall\, \phi \in S^1\,.$$

Our idea is to define an energy that measures how close γ is to a Reeb orbit. For this we use a Riemannian metric g on M. Since we are somewhat free to choose this metric we will do it in the best possible way. This means we choose an adapted Riemannian metric g on TM in the sense that

$$i_X g = \lambda\,.$$

In the forthcoming we will simply use the usual brackets $\langle\cdot,\cdot\rangle$ to denote the metric g. With this it follows

$$|X| = 1\,,\quad \langle W, X\rangle = 0\,, \forall\, W \in \xi\,.$$

Since the length of $i_{\gamma'}d\lambda$ depends on the parametrization of γ, we choose a parametrization by arc length so that

$$e := \gamma'$$

has unit length. Then γ is a Reeb orbit, if and only if

$$p := i_e\lambda = \langle X, e\rangle$$

satisfies $p^2 = 1$ for all points on γ, where the sign of p depends on the orientation of the curve. Let ν be the vector dual to $i_e d\lambda$. The vector field ν is normal along γ and vanishes exactly when γ is a Reeb orbit.

Let $s = s(p)$ be an arbitrary smooth function of p and denote $s' := \partial s/\partial p$. Then we make the following definition:

Definition 1. *Let (M, λ, g) be a contact manifold with adapted Riemannian metric g. For a closed curve $\gamma : S^1 \to M$, parametrized by arc length ϕ, and an arbitrary smooth function $s : [-1, 1] \to \mathbb{R}$ we define the* Reeb energy

$$E_s(\gamma) := \int_{S^1} s(p)\mu\,,$$

where $p = \lambda(\gamma')$ and μ denotes the induced volume form (line element) of γ on S^1.

Since S^1 is one-dimensional the induced metric $\gamma^* g$ on S^1 can be written in the form

$$\gamma^* g = \mu \otimes \mu\,. \tag{1}$$

Moreover, the second fundamental form A of γ satisfies

$$A = \nabla d\gamma = H \otimes \mu \otimes \mu\,, \tag{2}$$

where H denotes the mean curvature vector of γ and where here and in the following ∇ denotes the full Levi-Civita connection on S^1 resp. on bundles over S^1. The Levi-Civita connection on M will be denoted by D.

We want to compute the Euler-Lagrange equation for E_s. To this end let us assume that

$$\gamma : S^1 \times [0, T) \to M$$

is a smooth family of immersions $\gamma_t := \gamma(\cdot, t) : S^1 \to M$ that satisfy the evolution equation

$$\frac{d}{dt}\gamma(\phi, t) = V(\phi, t)\,, \tag{3}$$

where $V \in \Gamma(T^\perp S^1)$ is an arbitrary smooth section in the normal bundle.

Lemma 1. *If γ evolves under (3), then*

$$\frac{d}{dt}\gamma^* g = -2\langle V, H\rangle \gamma^* g\,, \tag{4}$$

$$\frac{d}{dt}\mu = -\langle V, H\rangle \mu\,. \tag{5}$$

Proof. We may compute this in local coordinates ϕ on S^1. From

$$(\gamma^* g)\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) = |\gamma'|^2 ,$$

with a prime denoting a partial derivative w.r.t. ϕ, we get

$$\begin{aligned}\left(\frac{d}{dt}\gamma^*\right)\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) &= 2\langle\gamma', V'\rangle \\ &= -2\langle\gamma'', V\rangle \\ &\overset{(1),(2)}{=} -2\gamma^*\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right)\langle H, V\rangle .\end{aligned}$$

which shows (4). (5) follows from this and $\mu = |\gamma'| d\phi$. □

Next we will compute evolution equations for various sections σ in vector bundles over S^1. In particular, we will consider those cases, where σ is a section in a vector bundle E_t which itself depends on time t. If for example ν_t is the normal vector field of the immersion $\gamma_t : S^1 \to M$, then ν_t is a section in $\gamma_t^{-1}TM$. In this case the mere computation of the partial derivative of ν_t w.r.t. t will be insufficient. To overcome this difficulty we just need to define a connection ∇ on $\gamma^{-1}TM$, where γ is now viewed as a map from the space-time manifold $S^1 \times [0, T)$ to M. A time derivative then becomes a covariant derivative in direction of $\frac{d}{dt}$, e.g. for a section $\nu \in \gamma^{-1}TM$ we have in local coordinates

$$\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha} , \quad \nabla_{\frac{d}{dt}}\nu = \left(\frac{\partial \nu^\alpha}{\partial t} + \Gamma^\alpha_{\beta\delta}\frac{\partial\gamma^\beta}{\partial t}\nu^\delta\right)\frac{\partial}{\partial y^\alpha} ,$$

where $\Gamma^\alpha_{\beta\delta}$ are the Christoffel symbols of the Levi-Civita connection on M and (y^α) are local coordinates on M. On the other hand, if σ is a section in a bundle E and E does not depend on t, then the covariant derivative $\nabla_{\frac{d}{dt}}\sigma$ coincides with $\frac{d}{dt}\sigma$.

Lemma 2. *If γ evolves under (3), then*

$$\nabla_{\frac{d}{dt}} e = \nabla_e V + \langle V, H\rangle e \tag{6}$$

$$\nabla_{\frac{d}{dt}} X = D_V X \tag{7}$$

$$\frac{d}{dt} p = -\langle \nu, V\rangle + \nabla_e\langle X, V\rangle + p\langle V, H\rangle . \tag{8}$$

Here, V and X are considered as a sections in $\gamma^{-1}TM$ and ∇ is the full connection on $\gamma^{-1}TM$.

Proof. Let us consider the section $d\gamma \in \Gamma(\gamma^{-1}TM \otimes T^*S^1)$. The bundle depends on t and we must use the covariant derivative $\nabla_{\frac{d}{dt}}$. Since in local coordinates $e|\gamma'| = \gamma'$ we get

$$\nabla_{\frac{d}{dt}} d\gamma = \nabla V = \nabla_e V \otimes \mu .$$

On the other hand $d\gamma = e \otimes \mu$ and (5) implies

$$\nabla_{\frac{d}{dt}} d\gamma = \left(\nabla_{\frac{d}{dt}} e - \langle V, H\rangle e\right) \otimes \mu .$$

Combining the last two equations gives

$$\nabla_{\frac{d}{dt}} e = \nabla_e V + \langle V, H\rangle e .$$

The evolution equation for X is clear. Since $\nabla_{\frac{d}{dt}} X = D_V X$ the evolution equation for p is

$$\begin{aligned}\frac{d}{dt}p &= \frac{d}{dt}\langle X, e\rangle \\ &= \langle\nabla_{\frac{d}{dt}} X, e\rangle + \langle X, \nabla_{\frac{d}{dt}} e\rangle \\ &= \langle D_V X, e\rangle + \langle X, \nabla_e V + \langle V, H\rangle e\rangle \\ &= \langle D_V X, e\rangle + \nabla_e\langle X, V\rangle - \langle D_e X, V\rangle + p\langle V, H\rangle \\ &= d\lambda(V, e) + \nabla_e\langle X, V\rangle + p\langle V, H\rangle \\ &= -\langle\nu, V\rangle + \nabla_e\langle X, V\rangle + p\langle V, H\rangle .\end{aligned}$$

□

From the previous results we obtain

$$\begin{aligned}\frac{d}{dt}(s\mu) &= \left(s'\frac{d}{dt}p - s\langle V,H\rangle\right)\mu \\ &= \big(s'(-\langle V,\nu\rangle + \nabla_e\langle X,V\rangle + p\langle V,H\rangle) - s\langle V,H\rangle\big)\mu \\ &= (\nabla_e(s'\langle X,V\rangle) - s''\langle X,V\rangle\nabla_e p - s'\langle V,\nu\rangle + (s'p-s)\langle V,H\rangle)\mu\end{aligned}$$

so that in view of $\nabla_e(s'\langle X,V\rangle)\mu = d(s'\langle X,V\rangle)$ and $\nabla_e p = D\lambda(e,e) + \langle X,H\rangle$ we obtain for any normal variation V:

$$\frac{d}{dt}\int_{S^1} s\mu = \int_{S^1}(-s''\langle X,V\rangle(D\lambda(e,e)+\langle X,H\rangle) - s'\langle V,\nu\rangle + (s'p-s)\langle V,H\rangle)\mu\,.$$

Consequently, if we take into account that the mean curvature vector H is normal, we have shown:

Theorem 1. *The negative L^2-gradient flow of $E_s(\gamma) = \int_{S^1} s\mu$ is*

$$\frac{d}{dt}\gamma = (s - s'p)H + s''(D\lambda(e,e) + \langle X,H\rangle)X^\perp + s'\nu\,, \tag{9}$$

where $X^\perp = X - pe$ is the normal part of X.

We call (9) the Reeb curve shortening flow for the potential s. For $s = 1$ we obtain the usual curve shortening flow. Let us consider the following special cases:

i) $s = -p$:

$$\frac{d}{dt}\gamma = -\nu\,. \tag{10}$$

ii) $s = 1 - cp$, for some constant c:

$$\frac{d}{dt}\gamma = H - c\nu\,. \tag{11}$$

iii) $s = 1 - p^2$:

$$\frac{d}{dt}\gamma = (1+p^2)H - 2(\langle X^\perp, H\rangle + D\lambda(e,e))X^\perp - 2p\nu\,. \tag{12}$$

Theorem 2 (Short-time existence)**.** *Let $\gamma_0 : S^1 \to M$ be a smooth immersion of a closed curve in a contact manifold (M,λ) with adapted Riemannian metric g and suppose that $s : [-1,1] \to \mathbb{R}$ is a smooth function that satisfies*

$$s(p) - ps'(p) > 0\,,\quad s(p) - ps'(p) + (1-p^2)s''(p) > 0\,,\quad \forall p \in [-1,1]\,. \tag{13}$$

Then the equation

$$\begin{aligned}\frac{d}{dt}\gamma &= (s - s'p)H + s''(D\lambda(e,e) + \langle X,H\rangle)X^\perp + s'\nu \\ \gamma(\cdot,0) &= \gamma_0\end{aligned}$$

is a quasilinear parabolic evolution equation and admits a smooth solution on a maximal time interval $[0,T)$ with $0 < T \le \infty$.

Proof. From $s - ps' > 0$ we obtain that (9) is a quasilinear system of second order. We want to prove that the operator on the RHS, i.e.

$$(s - s'p)H + s''(D\lambda(e,e) + \langle X,H\rangle)X^\perp + s'\nu$$

is elliptic. To this end we need to look at the linearized operator. Let $F : R \to M$ be an immersion of an r-dimensional manifold into a Riemannian manifold of dimension m. Then a second order differential operator $V[F] \in \Gamma(T^\perp R)$, assigning to F a section in the normal bundle $T^\perp R$ of R, is elliptic, if for any $\xi = \xi^i \partial/\partial x^i \in TR$, $\xi \neq 0$, the endomorphism $L \in \mathrm{End}(TM)$, locally given by

$$L^\alpha_\beta := \frac{\partial V^\alpha}{\partial F^\beta_{ij}} \xi^i \xi^j \,,$$

is non-degenerate on the normal bundle of R, i.e. if

$$g(LW, W) = g_{\alpha\gamma} L^\alpha_\beta W^\beta W^\gamma > 0 \,, \quad \forall W \in T^\perp R \,.$$

Here, F^β_{ij} is shorthand for $\partial^2 F^\beta / \partial x^i \partial x^j$ and $V = V^\alpha \partial/\partial y^\alpha$ is the expression for V in local coordinates $(y^\alpha)_{\alpha=1,\ldots,m}$ on M. In our case we have $F = \gamma$,

$$V^\alpha = (s - s'p) H^\alpha + s''(D\lambda(e,e) + \langle X, H \rangle)(X^\perp)^\alpha + s' \nu^\alpha$$

and $r = 1$, so that $F^\beta_{ij} = (\gamma^\beta)''$. From Gauß equation we know

$$|\gamma'|^2 H^\alpha = (\gamma^\alpha)'' - \frac{\langle \gamma', \gamma'' \rangle}{|\gamma'|^2} (\gamma^\alpha)' + \Gamma^\alpha_{\beta\delta} (\gamma^\beta)'(\gamma^\delta)' \,,$$

where $\Gamma^\alpha_{\beta\delta}$ are the Christoffel symbols of g. So

$$\frac{\partial H^\alpha}{\partial\big((\gamma^\beta)''\big)} = \frac{1}{|\gamma'|^2} \delta^\alpha_\beta - \frac{1}{|\gamma'|^4} g_{\beta\delta} (\gamma^\delta)'(\gamma^\alpha)' \,.$$

This implies

$$\begin{aligned}
\frac{\partial V^\alpha}{\partial\big((\gamma^\beta)''\big)} &= (s - ps') \left(\frac{1}{|\gamma'|^2} \delta^\alpha_\beta - \frac{1}{|\gamma'|^4} g_{\beta\delta} (\gamma^\delta)'(\gamma^\alpha)' \right) \\
&\quad + s'' g_{\delta\epsilon} (X^\perp)^\delta (X^\perp)^\alpha \left(\frac{1}{|\gamma'|^2} \delta^\epsilon_\beta - \frac{1}{|\gamma'|^4} g_{\beta\rho} (\gamma^\rho)'(\gamma^\epsilon)' \right) \\
&= \frac{1}{|\gamma'|^2} \left((s - ps') \delta^\alpha_\beta - e^\alpha e_\beta + s''(X^\perp)^\alpha (X^\perp)_\beta \right) \,,
\end{aligned}$$

where $e = e^\alpha \partial/\partial y^\alpha = \gamma^\alpha / |\gamma'|$ is the unit tangent vector and for example $e_\beta := g_{\beta\delta} e^\delta$. This implies that for an arbitrary $W \in TM$ we get

$$g(LW, W) = (s - ps')(|W|^2 - \langle W, e \rangle^2) + s'' \langle X^\perp, W \rangle^2 = (s - ps')|W^\perp|^2 + s'' \langle X^\perp, W \rangle^2 \,.$$

This clearly vanishes, if W is tangent to γ. These degeneracies in the symbol are due to the invariance of the equation under the diffeomorphism group of S^1 (for a general operator V as above, $g(LW, W)$ will vanish for $W \in TR$, causing as many degeneracies in the symbol as there are tangent directions on R. These degeneracies are called trivial). The symbol can only admit non-trivial degeneracies in the normal directions. Now suppose $W^\perp \neq 0$. We decompose $W^\perp$ into an orthogonal sum $W^\perp = W_1 + W_2$, where W_1 is parallel to $X^\perp$ and W_2 orthogonal to $X^\perp$. Then

$$\begin{aligned}
g(LW, W) &= (s - ps')|W^\perp|^2 + s'' \langle W_1, X^\perp \rangle^2 \\
&= (s - ps')|W_2|^2 + (s - ps')|W_1|^2 + s''|W_1|^2 \cdot |X^\perp|^2 \\
&= (s - ps')|W_2|^2 + (s - ps' + s''(1 - p^2))|W_1|^2 \,.
\end{aligned}$$

Since by assumption $s - ps' > 0$ and $s - ps' + s''(1 - p^2) > 0$ and either $W_1 \neq 0$ or $W_2 \neq 0$ we obtain

$$g(LW, W) > 0 \,.$$

So in this case the evolution equation is parabolic and the short-time existence of a smooth solution follows from the standard parabolic theory. □

Theorem 2 shows that (11) is always parabolic and that for the flow (12) we obtain a solution for a short time, if $p > 1/3$ on γ_0. Related to our flow equations are the following important questions:

I. What are the stationary solutions of the flow equation (9)?

II. Under what conditions can we prove long-time existence?

III. Under what conditions can we prove convergence?

IV. When do we have convergence to closed Reeb orbits?

Reeb orbits certainly are stationary solutions of (9), but the converse does not seem to hold. Instead from $V = 0$ with V as in (9) we obtain

Theorem 3. *The stationary solutions of the flows considered in Theorem 2 are characterized by the two equations*

$$(s - s'p)H^\nu + s''(D\lambda(e,e) + \langle X^\perp, H^\nu\rangle X^\perp = 0\,, \tag{14}$$

$$(s - s'p)(H - H^\nu) + s'\nu = 0\,, \tag{15}$$

where H^ν denotes the projection of H onto the orthogonal complement of ν. Moreover, if X is Killing w.r.t. g, then any stationary solution satisfies

$$p = c_1\,, \quad H = c_2\nu$$

for two constants c_1, c_2. In particular, if X is Killing and H vanishes in one point, then H vanishes everywhere. If ν vanishes in at least one point, then ν and H vanish everywhere and the stationary solution is a closed Reeb orbit that is also a closed geodesic w.r.t. g.

Proof. Equations (14) and (15) follow from $V = 0$ and

$$\langle X^\perp, \nu\rangle = (i_e d\lambda)(X) = -(i_X d\lambda)(e) = 0\,.$$

If X is Killing, then $D\lambda$ is skew, so that $D\lambda(e,e) = 0$. In this case we can multiply (14) with X and obtain from $|X^\perp|^2 = (1 - p^2)$ the relation

$$(s - s'p + s''(1 - p^2))\langle X, H\rangle = 0$$

and since $D_e p = D\lambda(e,e) + \langle X, H\rangle = \langle X, H\rangle$ we obtain from $s - s'p + s''(1 - p^2) > 0$ that $D_e p = 0$ and that p is constant. Then $(s - s'p)H^\nu = 0$ and $(s - s'p)H + s'\nu = 0$. By assumption $s - ps' > 0$ and since p is constant $s - ps'$ is constant as well. If H vanishes at one point, then either $c_2 = 0$ and H vanishes everywhere or $c_2 \neq 0$ and $\nu = 0$ at the same point. But $\nu = 0 \Leftrightarrow p^2 = 1$ so that by the constancy of p we must have $p^2 = 1$ everywhere, hence in this case $\nu \equiv 0$ and $H \equiv 0$. This proves that the stationary solution must be a either a geodesic (if H vanishes somewhere,) or a closed geodesic Reeb orbit, if ν vanishes somewhere. □

In general we do not expect that the flows considered in Theorem 2 admit long-time solutions. For example, if γ_0 is a closed curve close to some closed Reeb orbit on S^3 equipped with its standard contact structure, then this orbit - considered as a closed geodesic - is unstable under the curve shortening flow and certainly will not behave too good under our flow which can be considered as a perturbed curve shortening flow. Under additional assumptions, however, it is very likely that this flow can be used to generate closed Reeb orbits. We expect that long-time existence holds, if the second fundamental form of the evolving curves is uniformly bounded. This is the case for the curve shortening flow (and more generally for the mean curvature flow in arbitrary dimension and codimension). Most likely long-time existence should come with convergence, at least for subsequences which would be sufficient for our purposes.

2 The flow in Sasaki manifolds

The analysis of the flow equation simplifies significantly, if the ambient space is a Sasaki manifold. In the sequel we will therefore focus on this special situation and we will derive some general results.

2.1 Sasaki manifolds

A Sasaki manifold is a contact manifold (M, λ) with an adapted Riemannian metric $\langle \cdot, \cdot \rangle$ such that $D\lambda$ is skew (so that $d\lambda = 2D\lambda$) and such that $J \in \mathrm{End}(TM)$ defined by

$$JW := D_W X$$

satisfies

$$J^2 = -\mathrm{Id} + X \otimes \lambda = -\pi\,, \tag{16}$$

$$\langle JW, JZ \rangle = \langle W, Z \rangle - \lambda(W)\lambda(Z)\,, \quad \forall\, W, Z \in TM \tag{17}$$

and the integrability condition

$$(D_W J)(Z) = -\langle W, Z \rangle X + \lambda(Z) W\,, \quad \forall\, W, Z \in TM\,. \tag{18}$$

In particular, J defines a complex structure on ξ and the Reeb vector field X becomes a unit length Killing vector field so that Reeb orbits are also geodesics (clearly, the converse does not hold).

2.2 Curves in Sasaki manifolds

Let $\gamma : S^1 \to M$ be a regular closed curve in a Sasakian manifold and as before denote the positively oriented unit length tangent vector of γ by e. Let $\sharp : T^*M \to TM$ and $\flat : TM \to T^*M$ denote the (musical) isomorphisms induced by the metric g. Then

$$\nu = (i_e d\lambda)^\sharp = 2(i_e D\lambda)^\sharp = 2D_e X = 2Je\,. \tag{19}$$

Lemma 3. *The following relations hold on a Sasaki manifold:*

$$|\nu|^2 = 4(1 - p^2)\,, \quad \langle \nu, X \rangle = 0\,, \quad \langle \nu, e \rangle = 0\,, \tag{20}$$

$$dp = \langle X, H \rangle \mu = \langle X^\perp, H \rangle \mu\,, \tag{21}$$

$$dp^2 = -\langle \nu, JH \rangle \mu\,. \tag{22}$$

Proof. (20) follows from the definition of ν and p and from $J^2 = -\pi$. (21) follows from

$$\begin{aligned} dp(e) &= e\langle X, e \rangle \\ &= \langle D_e X, e \rangle + \langle X, D_e e \rangle \\ &= \langle Je, e \rangle + \langle X, H \rangle \\ &= \langle X, H \rangle\,. \end{aligned}$$

Finally (22) can be derived in the following way

$$\langle JH, \nu \rangle = 2\langle JH, Je \rangle = 2\langle H, e \rangle - 2\langle X, H \rangle \langle X, e \rangle = -2p\langle X, H \rangle \overset{(21)}{=} -\nabla_e p^2\,.$$

□

Lemma 4. *For $\nabla \nu \in \Gamma(\gamma^{-1}TM \otimes T^*S^1)$ holds*

$$\nabla \nu = 2(-X + pe + JH) \otimes \mu\,. \tag{23}$$

Proof. Since there is only one tangent direction, it suffices to prove

$$\nabla_e \nu = 2(-X + pe + JH)\,.$$

We compute

$$\begin{aligned} \nabla_e \nu &= 2\nabla_e (Je) \\ &= 2(D_e J)e + 2J(D_e e) \\ &\overset{(18)}{=} -2|e|^2 X + 2\lambda(e)e + 2J(A(e, e)) \\ &\overset{(2)}{=} 2(-X + pe + JH)\,. \end{aligned}$$

□

2.3 The evolution equations in Sasaki manifolds

In this section we derive some evolution equations in the special case, where the ambient space is a Sasaki manifold. The first observation is, that our basic evolution equation (9) simplifies a bit, since now $D\lambda(e,e)=0$ so that we have

$$\frac{d}{dt}\gamma = (s-ps')H + s''\langle X,H\rangle X^{\perp} + s'\nu\,. \tag{24}$$

Lemma 5. *If γ evolves by (24), then p satisfies the evolution equation*

$$\begin{aligned}\frac{d}{dt}p &= (s-ps'+(1-p^2)s'')\Delta p + (-2ps''+(1-p^2)s''')|\nabla p|^2\\ &\quad +p(s-ps')|H|^2 + (2ps'-s)\langle \nu,H\rangle - 4(1-p^2)s'\,. \end{aligned} \tag{25}$$

Proof. From (8) we get

$$\frac{d}{dt}p = -\langle \nu,V\rangle + \nabla_e\langle X,V\rangle + p\langle V,H\rangle$$

with

$$V = (s-ps')H + s''\langle X,H\rangle X^{\perp} + s'\nu\,.$$

Since

$$\begin{aligned}\langle X,V\rangle &= (s-ps')\langle X,H\rangle + s''\langle X,H\rangle |X^{\perp}|^2\\ &\overset{(21)}{=} (s-ps'+(1-p^2)s'')\nabla_e p\end{aligned}$$

we get

$$\nabla_e\langle X,V\rangle = (s-ps'+(1-p^2)s'')\Delta p + (-3ps''+(1-p^2)s''')|\nabla p|^2\,.$$

Moreover,

$$p\langle V,H\rangle = p(s-ps')|H|^2 + ps''\langle X,H\rangle^2 + ps'\langle \nu,H\rangle$$

and

$$\langle \nu,V\rangle = (s-ps')\langle \nu,H\rangle + s'|\nu|^2\,.$$

Then the result follows from $|\nu|^2 = 4(1-p^2)$ and $\langle X,H\rangle^2 = |\nabla p|^2$. □

We finish this paper with a C^1-estimate for the Reeb flows considered in Theorem 2

Lemma 6. *Suppose γ is a closed curve in a Sasaki manifold that evolves by (9) for some function s satisfying (13) for all $t\in[0,T)$. Further suppose that s is a function for which $s\le 0$ implies $p>0$. If $s<0$ on γ_0, then this holds on γ_t for all $t\in[0,T)$.*

Proof. Suppose this does not hold and let $t_0\in(0,T)$ be the first time where $s=0$ at some point on γ_{t_0}. Then we have $s\le 0$ on $[0,t_0]$ and $s<0$ on $[0,t_0)$. From $s\le 0$ we obtain $p>0$ and then $s-ps'>0$ implies $s'<0$ on $[0,t_0]$. From the evolution equation for p we obtain

$$\begin{aligned}\frac{d}{dt}s &= s'\frac{d}{dt}p\\ &= (s-ps'+(1-p^2)s'')\Delta s\\ &\quad +[s'(-2ps''+(1-p^2)s''') - s''(s-ps'+(1-p^2)s'')]|\nabla p|^2\\ &\quad +p(s-ps')s'|H|^2 + (2ps'-s)s'\langle \nu,H\rangle - (s')^2|\nu|^2\,.\end{aligned} \tag{26}$$

Now by Schwarz' inequality we have for $\epsilon>0$

$$(2ps'-s)s'\langle \nu,H\rangle \le -s'\left(\epsilon|H|^2 + \frac{(2ps'-s)^2}{4\epsilon}|\nu|^2\right)\,.$$

Since $\epsilon := p(s-ps')>0$ on $[0,t_0]$ we obtain

$$(2ps'-s)s'\langle \nu,H\rangle \le -s'p(s-ps')|H|^2 - \frac{s'(2ps'-s)^2}{4p(s-ps')}|\nu|^2\,.$$

This implies

$$\begin{aligned}\frac{d}{dt}s &\le (s-ps'+(1-p^2)s'')\Delta s\\ &\quad +[s'(-2ps''+(1-p^2)s''')-s''(s-ps'+(1-p^2)s'')]|\nabla p|^2\\ &\quad -\frac{s'|\nu|^2}{4p(s-ps')}\,s^2\end{aligned}$$

for all $t\in[0,t_0]$. Since $t_0<T$ the quantity $\left|\frac{s'|\nu|^2}{4p(s-ps')}\,s\right|$ is certainly bounded from above by some positive constant K on $[0,t_0]$. So on $[0,t_0]$ we conclude

$$\begin{aligned}\frac{d}{dt}s &\le (s-ps'+(1-p^2)s'')\Delta s\\ &\quad +[s'(-2ps''+(1-p^2)s''')-s''(s-ps'+(1-p^2)s'')]|\nabla p|^2+Ks\,.\end{aligned}$$

We want to apply the strong parabolic maximum principle to the function $\tilde{s}:=se^{-Kt}$. Taking into account that $s-ps'+(1-p^2)s''>0$ and observing that $\nabla s=s'\nabla p=0$ implies $\nabla p=0$ (since $s'<0$), we obtain

$$se^{-Kt}<0$$

for all $t\in[0,t_0]$. This is a contradiction to the assumption that s vanishes for the first time at some point on γ_{t_0}. □

Example. Suppose there exists a constant $\epsilon>0$ such that $p>\epsilon$ on γ_0. Then the function $s=1-p/\epsilon$ satisfies all assumptions in Lemma 6 and $p>\epsilon$ for all $t\in[0,T)$.

The condition $p>0$ can be interpreted in the sense that γ is a graph over a Reeb orbit. In mean curvature flow, long-time existence and convergence will often be implied by such a graphical condition (see e.g. [2], [7]). In analogy to this situation we expect that a lower bound for p might be useful to get upper bounds for the second fundamental form such that long-time existence and perhaps convergence would follow. This will be the subject in a forthcoming paper.

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