Symmetric hypersurfaces in Riemannian manifolds contracting to Lie-groups by their mean curvature

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Abstract. This paper concerns the deformation by mean curvature of hypersurfaces $\widetilde{M}$ in Riemannian spaces $\widetilde{N}$ that are invariant under a subgroup of the isometry-group on $\widetilde{N}$. We show that the hypersurfaces contract to this subgroup, if the cross-section satisfies a strong convexity assumption.

In an earlier paper [S2] the author considered the case of hypersurfaces in euclidean space that are symmetric w.r.t rotations in a 2-plane. In this paper we transfer the techniques to the problem of hypersurfaces in Riemannian spaces that are invariant under isometry-actions by Lie-groups.

Let $\widetilde{M}^m$ be a hypersurface smoothly immersed in a Riemannian manifold $\widetilde{N}^{m+1}$ and let $\widetilde{M}_0 := \widetilde{M}^m$ be given locally by some diffeomorphism

$$\widetilde{F}_0 : \ \widetilde{U} \subset I\!R^m \longrightarrow \widetilde{F}_0(\widetilde{U}) \subset \widetilde{M}_0 \subset \widetilde{N}^{m+1}$$

Then we want to find a family $\widetilde{F}(\cdot, t)$ of diffeomorphisms corresponding to hypersurfaces

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$\widetilde{M}_t$ that solve the evolution equation

$$\text{(1)} \qquad \frac{d}{dt}\widetilde{F}(\widetilde{x},t) = \overrightarrow{H}(\widetilde{x},t) \quad , \quad \widetilde{x} \in \widetilde{U}$$

$$\widetilde{F}(\cdot,0) = \widetilde{F}_0$$

Here $\overrightarrow{H}(\widetilde{x},t) = -\widetilde{H}(\widetilde{\nu})\widetilde{\nu}$ is the inward pointing mean curvature vector on $\widetilde{M}_t$.

Gage and Hamilton [GH] investigated the evolution process for curves in $I\!R^2$ and proved that any convex, embedded, closed curve contracts to a single point; later this result was extended by Grayson [Gr] to the case of any embedded curve in $I\!R^2$. An important result concerning convex hypersurfaces in Riemannian spaces is due to Huisken [H2] who used the parametric version above and proved that closed hypersurfaces contract to a single point if they satisfy a strong convexity assumption. He also investigated the singularity behaviour [H3], as was also done by Altschuler, Angenent, Grayson [Alt], [AG], [Ang1] e.a. If the hypersurface is a level set of a suitable function, one can define viscosity solutions. This was established by Chen, Giga, Goto [CGG], Evans, Spruck [ES] and Ilmanen [Ilm1]. Ilmanen also clarified the relation between the level set approach and the varifold approach [Ilm2]. Recently Luckhaus and Sturzenhecker [LS] introduced a weak notion in the BV -setting and Jost [J] gave another weak formulation of the mean curvature flow enabling the study of three or more phase problems.

In this paper we want to study the behaviour of hypersurfaces that are invariant under the action of a Lie-group G . To make this precise we assume:

Let G^k be a k-dimensional Lie group that acts smoothly, freely, and properly on $\widetilde{N}^{n+1+k}$.

This action defines a quotient space $N := {}^{\widetilde{N}}/_{G}$. The class of a point p will be denoted by $[p]$. It is a well known result that there is exactly one structure of a differentiable manifold on N such that the projection $\pi : \widetilde{N} \to N, \quad \pi(p) := [p]$ gives a submersion (see e.g. [GHL], chpt. 1 and 2). If in addition G acts by isometries on $\widetilde{N}$, then π becomes to a

Riemannian submersion. Each $[p]$ is a k-dimensional smooth submanifold in $\widetilde{N}$. Given any two points $p_1,\ p_2 \in [p]$, there is exactly one $\Psi \in G$ such that $\Psi(p_1) = \Psi(p_2)$ and since Ψ is an isometry, the differential $D\Psi$ maps $T_{p_1}[p]$ to $T_{p_2}[p]$ and $(T_{p_1}[p])^\perp$ to $(T_{p_2}[p])^\perp$. The quotient of the normal bundle $(T[p])^\perp$ then gives the tangent space $T_{[p]}N$. So for each $q \in [p]$ we have the decomposition

$$T_q\widetilde{N} = T_q[p] \oplus T_{[p]}N = T_{Id}G \oplus T_{[p]}N$$

If $\widetilde{M}^{n+k} \subset \widetilde{N}^{n+1+k}$ is a smooth hypersurface in $\widetilde{N}$, invariant under the action of G , then $M := {}^{\widetilde{M}}/_{G}$ is a hypersurface in N and we can decompose the tangent space of $\widetilde{M}$

$$T_q\widetilde{M} = T_{Id}G \oplus T_{[p]}M$$

for each $q \in [p]$.

Now let ν be a normal vector on $M \subset N$. Then the horizontal lift $\widetilde{\nu}$ gives a normal vector field along a fiber $[p] \in \widetilde{M}$; so if $\widetilde{A}$ is the second fundamental form on $\widetilde{M}$ for a normal vector $\widetilde{\nu}$, we obtain n eigenvalues $\lambda_1, \ldots, \lambda_n$ with $Eig(\lambda_i) \subset T_{[p]}M$ and k eigenvalues $\lambda_{n+1}, \ldots, \lambda_{n+k}$ with $Eig(\lambda_j) \subset T_{Id}G$. Therefore we get

$$(*) \qquad \overrightarrow{H} = \overrightarrow{H}_M - \lambda\widetilde{\nu}$$

, where $\overrightarrow{H} = -\widetilde{H}\widetilde{\nu}$ is the mean curvature vector on $\widetilde{M}$, $\overrightarrow{H}_M$ the mean curvature vector on $M \subset N$ and $\lambda := \lambda_{n+1} + \ldots + \lambda_{n+k}$.

To give another expression for λ , we calculate as follows:

Let K be a smooth submanifold in $\widetilde{N}$ with $codim(K) = l$ and let the metric and connection on $\widetilde{N}$ be denoted by $\widetilde{g},\ \widetilde{\nabla}$ respectively. The second fundamental form A on K is the section in $(TK)^{\perp *} \otimes TK^* \otimes TK^*$ with $A(\widetilde{\nu}, u, v) = -\widetilde{g}(\widetilde{\nabla}_u v, \widetilde{\nu})$. Let $\widetilde{\nu}_\alpha$, $\alpha = 1, \ldots, l$ be a local basis on $(TK)^\perp$, $A_\alpha := A(\widetilde{\nu}_\alpha, \cdot, \cdot)$, $g^\perp{}_{\alpha\beta} := \widetilde{g}(\widetilde{\nu}_\alpha, \widetilde{\nu}_\beta)$ and let $(g^{\perp})^{\alpha\beta}$ be inverse to ${g^{\perp}}_{\alpha\beta}$. The mean curvature vector field on K is the canonical section in $(TK)^{\perp}$ given by

$$\overrightarrow{H}_K := -(g^{\perp})^{\alpha\beta} trace(A_\alpha)\widetilde{\nu}_\beta$$

Now let $K=[p]$, $\widetilde{\nu}$ be a normal vector field on $\widetilde{M}$ and $e_1,\ldots,e_{k+n}$ be a basis for $T_q\widetilde{M}$ of orthonormal vectors such that A becomes diagonal and furtheron assume that $e_1,\ldots,e_n$ form a local basis of $T_{[p]}M$ and $e_{n+1},\ldots,e_{n+k}$ a local basis of $T_{Id}G$. Then we see that

$$\lambda = \sum_{i=n+1}^{n+k} \lambda_i = -\sum_{i=n+1}^{n+k} \widetilde{g}(\widetilde{\nabla}_{e_i} e_i, \widetilde{\nu}) = -\widetilde{g}(\overrightarrow{H}_{[p]}, \widetilde{\nu})$$

and $(*)$ takes the form

$$(**) \qquad \overrightarrow{H} = \overrightarrow{H}_M + \widetilde{g}(\overrightarrow{H}_{[p]}, \widetilde{\nu})\widetilde{\nu}$$

, i.e. $-\lambda\widetilde{\nu}$ is the projection of $\overrightarrow{H}_{[p]}$ onto $T\widetilde{M}^{\perp}$.

Since equation (1) is isotropic, we conclude that the solutions of (1) must stay invariant under the G-action as long as a smooth solution of (1) exists. This means that the behaviour of $\widetilde{M_t}$ is totally determined by the shape of the cross-sections $M_t := {}^{\widetilde{M_t}}/_{G}$. For that reason we can investigate the solutions of the following evolution process instead of (1) :

$$(2) \qquad \frac{d}{dt}F(x,t) = \overrightarrow{H}(x,t) + \bar{g}(\overrightarrow{H}_{[p]}(x,t), \nu(x,t))\nu(x,t)$$

$$F(\cdot,0) = F_0$$

$$F(x,t) \in N$$

, where $\bar{g}$ is the metric on N induced by $\widetilde{g}$.

Equation (2) is a quasilinear parabolic system, since this is true for (1) and the extra term in (2) gives only first order terms. Therefore a unique solution exists for a short time, if the cross-section is compact but more important is the fact that any solution of (2) gives a solution of (1) and the opposite is true as well. Equation (2) is (in general) an anisotropic flow and the extra term in (2) can be interpreted as the normal component of an exterior force given by the field $\overrightarrow{H}_{[p]}$.

We write $g = \{g_{ij}\}$, Γ_{ij}^k , $A = \{h_{ij}\}$ for the induced metric, Levi-Civita connection and the second fundamental form on M_t . Double latin indices are summed from 1 to n and greek ones from 0 to n . Expressions on N will be denoted by a bar e.g. $\bar{R}_{\alpha\beta\gamma\delta}$ for the curvature tensor on N. $i(N)$ is the injectivity radius and $\bar{K}(X,Y)$ the sectional curvature on N . Then the equation of O'Neill [O'N] takes the form

$$\bar{K}(X,Y) = \widetilde{K}(\widetilde{X},\widetilde{Y}) + \frac{3}{4}|[\widetilde{X},\widetilde{Y}]^v|^2 \geq \widetilde{K}(\widetilde{X},\widetilde{Y})$$

, where $\widetilde{X}, \widetilde{Y}$ are the horizontal lifts of the orthonormal vector fields X and Y and ^{v} denotes the vertical component. That means in particular that curvature bounds on $\widetilde{N}$ will not imply the same bounds on N. Unfortunately this forces us to use curvature bounds for each of the spaces N and $\widetilde{N}$. In the case where the interior of M_0 is contained in a compact subspace of N and M_0 is convex enough, this will not yield much complications since M_t will stay in the interior of M_0 and if we replace N by $int(M_0)$ the compactness will give curvature bounds.

In the following sections we make the assumption that $n \geq 2$ and that N^{n+1} is a smooth, complete Riemannian manifold without boundary satisfying the conditions

There are nonnegative constants $K_1,\ K_2,\ L,\ \widetilde{K}_1,\ \widetilde{K}_2,\ \widetilde{L}$ with

(1.0.0)
$$-K_1 \leq \bar{K}(X,Y) \leq K_2$$
$$|\bar{\nabla}\bar{R}|^2 \leq L^2$$
$$i(N) > 0$$

(1.0.1)
$$-\widetilde{K}_1 \leq \widetilde{K}(\widetilde{X},\widetilde{Y}) \leq \widetilde{K}_2$$
$$|\widetilde{\nabla}\widetilde{R}|^2 \leq \widetilde{L}^2$$

There is a constant $B \geq 0$, such that for all $p \in N$:

(1.0.2)
$$|\overrightarrow{H}_{[p]}| \leq B$$

$$|\bar{\nabla}\overrightarrow{H}_{[p]}| = \max_{|u|=1} |\bar{\nabla}_u \overrightarrow{H}_{[p]}| \leq B$$

$$|\bar{\nabla}^2\overrightarrow{H}_{[p]}| = \max_{|u|,\ |v|=1} |\bar{\nabla}_u \bar{\nabla}_v \overrightarrow{H}_{[p]}| \leq B$$

The bound for the sectional curvature implies a bound for the full curvature tensor, i.e. there is a nonnegative constant c_n , such that

$$\bar{R}(X,Y,Z,W) \leq c_n \max(K_1,K_2)|X||Y||Z||W|$$

for all vectorfields $X,\ Y,\ Z,\ W$.

Let $|A|^2 = g^{ik}g^{jl}h_{ij}h_{kl}$

$C := g^{ij}g^{kl}g^{mn}h_{ik}h_{jm}h_{ln}$

$Z := HC - |A|^4$

$Q^2 := |\nabla_i h_{kl} H - \nabla_i H h_{kl}|^2 = |\nabla A|^2 H^2 + |\nabla H|^2 |A|^2 - H < \nabla_i |A|^2, \nabla_i H >$

and let $w = \{w_i\}$ be the projection of $\bar{R}ic(\nu,\cdot)$ on M_t , i.e.

the vector with components $w_i = \bar{R}_{0li}{}^l$.

Then we can state our main theorem:

Theorem 1.0 : *Let M_0 be a closed, connected hypersurface smoothly immersed into N and assume that for $0 < \epsilon < \frac{1}{n}$ we have on M_0:*

$$(a_1) \qquad H > n\sqrt{K_1} + lB$$

$$(a_2) \qquad H + \lambda > (n+k)\sqrt{\widetilde{K}_1}, \quad if\ B \neq 0$$

with $l \geq 0$ satisfying the inequality

$$(b) \qquad B[(\frac{l^2 B}{n} - 2\sqrt{n})lB - n(1 + c_n \max(K_1, K_2))] \geq 0$$

and assume there is a constant k with

$$(c) \qquad h_{ij} \geq \epsilon H g_{ij} + \frac{n(1-n\epsilon)}{H} K_1 g_{ij} + \frac{n^2}{H^2} L g_{ij} + kB g_{ij}$$

where $k = 0$ if $B = 0$ and if $B > 0$

$$(d) \qquad nkB^2(k^2B - 4) - 2\frac{n}{k}(K_1 + 2\frac{L}{kB}) - (1 + c_n \max(K_1, K_2))(2B + \frac{1}{k^2 B}(K_1 + 2\frac{L}{kB})) \geq 0$$

Then (2) has a smooth solution M_t for finite time $[0, T)$ and the solution contracts to a single point $p \in N$ for $t \to T$. If we use homothetic expansions of normal coordinates around p such that the total area of the rescaled surfaces are constant in time, then the surfaces converge to a sphere in the C^∞-topology.

Corollary 1.1 : *Assume that $\widetilde{M}$ is an immersion of a hypersurface in $\widetilde{N}$, such that the cross-sections M and N satisfy the conditions of theorem 1.0. Then in* *the same time the hypersurfaces* $\widetilde{M}_t$ *contract under the mean curvature flow to a single fiber* $[p]$ *diffeomorphic to* G.

Remarks:

If N has a boundary ∂N and if the mean curvature on the boundary with respect to the inner normal satisfies the inequality

$$\inf_{\partial N} H \geq -n\sqrt{K_1} - lB$$

, then it follows from the strong elliptic maximum principle that M_t cannot touch ∂N and theorem 1.0 will hold in this case too.

The constant B in condition (1.0.2) can be replaced by three different constants, but this will complicate the calculations and make them much more complex.

In the case $B = 0$ theorem 1.0 takes exactly the same form as in [H2]. If in addition N is as in [H2] and if we set $\widetilde{N} := I\!R \times N$, $G := I\!R$ and define the action of G on $\widetilde{N}$ simply by translations along the real component it follows $B = 0$. So theorem 1.0 can be seen as an extension of the results in [H2].

The proof does not work for $n = 1$ since we make use of the Codazzi-equations which are worthless in case $n = 1$. We expect that a comparable result will also hold in the one-dimensional case since e.g. Ishimura [I] could show that two-dimensional tori contract to a circle, if the cross-section is a curve with suitable convexity properties.

In general the result is false for hypersurfaces "very near" to a symmetric hypersurface. E.g., if one investigates a rotationally symmetric surface in $I\!R^3$, subjected to small deviations in the diameter then this surface will contract faster at those cross-sections with smaller diameters. This was proved by [AAG] and the author [S1].

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2. Evolution equations for various curvature flows:

I. The general case

In the following section we want to derive some evolution equations for various curvature flows. For this purpose we assume that $F(x,t)$ is the immersion of a hypersurface M_t into a Riemannian manifold N and that $F(x,t)$ is a solution of the differential equation

$$(+) \qquad \frac{d}{dt}F(x,t) = -f(x,t)\nu(x,t)$$

, where f is a smooth speed-function and ν a normal vector field on M_t.

For a fixed time t we choose a local field of frames $e_0,\ e_1,\dots,e_n$ for T_pN , such that for the restriction on M_t we have $e_0 = \nu$, $e_i = \frac{\partial F}{\partial x_i}$.

Then we have the well known relations:

<u>**Lemma 2.1**</u> :

(a) $$g_{ij} = \bar{g}_{\alpha\beta}\frac{\partial F^\alpha}{\partial x_i}\frac{\partial F^\beta}{\partial x_j}$$

(b) $$h_{ij} = \bar{g}_{\alpha\beta}\frac{\partial F^\alpha}{\partial x_i}\frac{\partial \nu^\beta}{\partial x_j} + \bar{\Gamma}^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x_i}\frac{\partial F^\beta}{\partial x_j}\nu^\sigma\bar{g}_{\alpha\beta}$$

(c) $$\frac{\partial^2 F^\alpha}{\partial x_i\partial x_j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x_k} + \bar{\Gamma}^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x_i}\frac{\partial F^\sigma}{\partial x_j} = -h_{ij}\nu^\alpha$$

(d) $$\frac{\partial \nu^\alpha}{\partial x_j} + \bar{\Gamma}^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x_j}\nu^\sigma = h_{jl}g^{lm}\frac{\partial F^\alpha}{\partial x_m}$$

(e) $$R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}$$

(f) $$\nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{0ijk}$$

(g) $$\Delta h_{ij} = \nabla_i\nabla_j H + Hh_{il}h^l{}_j - |A|^2 h_{ij} + H\bar{R}_{0i0j} - h_{ij}\bar{R}_{0l0}{}^l + h_{jl}\bar{R}^l{}_{mi}{}^m + h_{il}\bar{R}^l{}_{mj}{}^m - 2h_{lm}\bar{R}^l{}_i{}^m{}_j + \bar{\nabla}_j\bar{R}_{0li}{}^l + \bar{\nabla}_l\bar{R}_{0ij}{}^l$$

(h) $$\Delta|A|^2 = 2 < h_{ij}, \nabla_i\nabla_j H > +2|\nabla A|^2 + 2Z + 2Hh^{ij}\bar{R}_{0i0j} - 2|A|^2\bar{R}_{0l0}{}^l + 4h^{ij}(h_{jl}\bar{R}^l{}_{mi}{}^m - h^{lm}\bar{R}_{limj}) + 2h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^l + \bar{\nabla}_l\bar{R}_{0ij}{}^l)$$

To calculate the evolution equations we can assume that at a fixed point (p,t) we have introduced normal coordinates for M_t and N and that $\nu^\alpha = \delta^\alpha_0$, $\frac{\partial F^\alpha}{\partial x_i} = \delta^\alpha_i$, such that all Christoffel-symbols $\bar{\Gamma}^{\gamma}_{\alpha\beta}$ vanish and that in particular $\bar{g}_{\alpha\beta,\gamma} = 0$ for all α, β, γ.

Then simple calculations show that

Lemma 2.2 : (a) $\frac{d}{dt} g_{ij} = -2fh_{ij}$

(b) $\frac{d}{dt} d\mu = -Hf d\mu$

Lemma 2.3 : $\frac{d}{dt}\nu = \nabla f$

Lemma 2.4 : $\frac{d}{dt} h_{ij} = \nabla_i\nabla_j f - fh_{il}g^{lm}h_{mj} + f\bar{R}_{0i0j}$

Lemma 2.5 : $\frac{d}{dt} H = \Delta f + f(|A|^2 + \bar{R}ic(\nu,\nu))$

Lemma 2.6 : $\frac{d}{dt}|A|^2 = 2 < h_{ij}, \nabla_i\nabla_j f > +2fC + 2f < h_{ij}, \bar{R}_{0i0j} >$

II. The evolution equations in the case $f = H + \lambda$:

We will need the following tensors which take their values on M:

Definition : (a) $a_{ij} := \bar{g}(e_i, \bar{\nabla}_{e_j}\overrightarrow{H}_{[p]})$

(b) $b_{ij} := \bar{g}(\nu, \bar{\nabla}_{e_i}\bar{\nabla}_{e_j}\overrightarrow{H}_{[p]})$

(c) $V_{ij} := -h_{il}g^{lm}a_{mj} - h_{jl}g^{lm}a_{mi} - b_{ij} - < \bar{R}_{0jli}, \overrightarrow{H}^0_l > +\lambda\bar{R}_{0i0j}$

(d) $V := trace(V_{ij}) = g^{ij}V_{ij}$

, where $\overrightarrow{H}^0 := \bar{g}(\overrightarrow{H}_{[p]}, \frac{\partial F}{\partial x_i})g^{ij}\frac{\partial F}{\partial x_j}$ is the projection of $\overrightarrow{H}_{[p]}$ onto M.

(Note that $e_i(x) = \frac{\partial F}{\partial x_i}(x)$)

Lemma 2.7 : (a) $\lambda = -\bar{g}(\nu, \overrightarrow{H}_{[p]})$

(b) $\nabla_i\lambda = -h_{il}g^{lm}\bar{g}(\frac{\partial F}{\partial x_m}, \overrightarrow{H}_{[p]}) - \bar{g}(\nu, \bar{\nabla}_{e_i}\overrightarrow{H}_{[p]})$

(c) $\nabla_i\nabla_j\lambda = V_{ij} - < \nabla_l h_{ij}, \overrightarrow{H}^0_l > -\lambda\bar{R}_{0i0j} - \lambda h_{im}g^{mn}h_{nj}$

$$(d) \qquad \Delta\lambda = - < \nabla_l H, \overrightarrow{H}_l^0 > + V - \lambda(|A|^2 + \bar{R}ic(\nu,\nu))$$

Proof: (a) is due to $(**)$ in section 1. To prove (b), we extend λ in a neighbourhood of U around p in N. The best way to do this, is to extend the normal vector field ν by a vector field e_0. We calculate

$$\nabla_i \lambda = -\bar{g}(\bar{\nabla}_{e_i} e_0, \overrightarrow{H}_{[p]}) - \bar{g}(\nu, \bar{\nabla}_{e_i} \overrightarrow{H}_{[p]})$$

and since

$$\begin{aligned}\bar{\nabla}_{e_i} e_0(x) = \frac{\partial F^\alpha}{\partial x_i} \bar{\nabla}_\alpha \nu^\beta \frac{\partial}{\partial y_\beta} &= \frac{\partial F^\alpha}{\partial x_i}((\frac{\partial \nu^\beta}{\partial y_\alpha} + \nu^\gamma \bar{\Gamma}^\beta_{\alpha\gamma})\frac{\partial}{\partial y_\beta}) \\ &= h_{il} g^{lm} \frac{\partial F}{\partial x_m}(x)\end{aligned}$$

this is independent of the extension and (b) follows. For (c) we use normal coordinates around p for N and M. Then all Christoffel-symbols vanish and in particular: $\nabla_i h_{jk} = h_{jk,i}$. So we get

$$\begin{aligned}\nabla_i \nabla_j \lambda = &- \bar{g}(\bar{\nabla}_{e_i} \bar{\nabla}_{e_j} e_0, \overrightarrow{H}_{[p]}) - \bar{g}(\bar{\nabla}_{e_i} e_0, \bar{\nabla}_{e_j} \overrightarrow{H}_{[p]}) \\ &- \bar{g}(\bar{\nabla}_{e_j} e_0, \bar{\nabla}_{e_i} \overrightarrow{H}_{[p]}) - \bar{g}(\nu, \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} \overrightarrow{H}_{[p]})\end{aligned}$$

and we calculate

$$\begin{aligned}\bar{\nabla}_{e_i} \bar{\nabla}_{e_j} e_0(x) &= (h_{jl} g^{lm} \frac{\partial F^\beta}{\partial x_m})_{,i} \frac{\partial}{\partial y_\beta} \\ &= \nabla_i h_{jl} g^{lm} \frac{\partial F}{\partial x_m} + h_{jl} g^{lm} \frac{\partial^2 F^\beta}{\partial x_i \partial x_m} \frac{\partial}{\partial y_\beta}\end{aligned}$$

If we use lemma 2.1 (c) and (f) we get

$$\bar{\nabla}_{e_i} \bar{\nabla}_{e_j} e_0(x) = (\nabla_l h_{ij} + \bar{R}_{0jli}) g^{lm} \frac{\partial F}{\partial x_m} - h_{jl} g^{lm} h_{im} \nu$$

and finally

$$\begin{aligned}\nabla_i \nabla_j \lambda = &- (\nabla_l h_{ij} + \bar{R}_{0jli}) g^{lm} \bar{g}(\frac{\partial F}{\partial x_m}, \overrightarrow{H}_{[p]}) + h_{jl} g^{lm} h_{mi} \bar{g}(\nu, \overrightarrow{H}_{[p]}) \\ &- h_{il} g^{lm} \bar{g}(\frac{\partial F}{\partial x_m}, \bar{\nabla}_{e_j} \overrightarrow{H}_{[p]}) - h_{jl} g^{lm} \bar{g}(\frac{\partial F}{\partial x_m}, \bar{\nabla}_{e_j} \overrightarrow{H}_{[p]}) \\ &- \bar{g}(\nu, \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} \overrightarrow{H}_{[p]}) \\ &= V_{ij} - < \nabla_l h_{ij}, \overrightarrow{H}_l^0 > - \lambda \bar{R}_{0i0j} - \lambda h_{im} g^{mn} h_{nj}\end{aligned}$$

and (c), (d) are obvious.

With lemmata 2.2, 2.3, 2.4, 2.5, 2.6 and with 2.1 (g), (h) and 2.7 (c), (d) we obtain the evolution equations:

Lemma 2.8 : (a) $\frac{d}{dt}g_{ij} = -2(H+\lambda)h_{ij}$

(b) $\frac{d}{dt}\nu = \nabla(H+\lambda)$

(c) $\frac{d}{dt}h_{ij} = \Delta h_{ij} - <\nabla_l h_{ij}, \overrightarrow{H}_l^0> -2(H+\lambda)h_{im}g^{mn}h_{nj}$

$+(|A|^2 + \bar{R}ic(\nu,\nu))h_{ij} - h_{jl}\bar{R}^l{}_{mi}{}^m - h_{il}\bar{R}^l{}_{mj}{}^m + 2h_{lm}\bar{R}^l{}_i{}^m{}_j$

$-\bar{\nabla}_j\bar{R}_{0li}{}^l - \bar{\nabla}_l\bar{R}_{0ij}{}^l + V_{ij}$

(d) $\frac{d}{dt}H = \Delta H - <\nabla_l H, \overrightarrow{H}_l^0> +H(|A|^2 + \bar{R}ic(\nu,\nu)) + V$

(e) $\frac{d}{dt}|A|^2 = \Delta|A|^2 - <\nabla_l|A|^2, \overrightarrow{H}_l^0> -2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{R}ic(\nu,\nu))$

$-4(h^{ij}h_j{}^m\bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj})$

$+2h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^l - \bar{\nabla}_l\bar{R}_{0ij}{}^l) + 2 <h_{ij}, V_{ij}>$

Lemma 2.9 : (a) $\frac{d}{dt}(\frac{1}{H}) = \Delta\frac{1}{H} - \frac{2}{H^3}|\nabla H|^2 - <\nabla\frac{1}{H}, \overrightarrow{H}^0> -\frac{1}{H}(|A|^2 + \bar{R}ic(\nu,\nu)) - \frac{V}{H^2}$

(b) $\frac{d}{dt}(\frac{1}{H^2}) = \Delta\frac{1}{H^2} - \frac{6}{H^4}|\nabla H|^2 - <\nabla\frac{1}{H^2}, \overrightarrow{H}^0> -\frac{2}{H^2}(|A|^2 + \bar{R}ic(\nu,\nu))$

$-2\frac{V}{H^3}$

We will need the following tensor:

$$M_{ij} := h_{ij} - \epsilon H g_{ij} - n(1-n\epsilon)K_1\frac{1}{H}g_{ij} - \frac{n^2L}{H^2}g_{ij} - kBg_{ij}$$

A simple calculation gives

Lemma 2.10 : $\frac{d}{dt}M_{ij} = \Delta M_{ij} - <\nabla_l M_{ij}, \overrightarrow{H}_l^0> -2(H+\lambda)h_{im}g^{mn}h_{nj}$

$$+(|A|^2 + \bar{R}ic(\nu,\nu))h_{ij} - h_{jl}\bar{R}^l{}_{mi}{}^m - h_{il}\bar{R}^l{}_{mj}{}^m + 2h_{lm}\bar{R}^l{}_i{}^m{}_j$$

$$-\bar{\nabla}_j \bar{R}_{0li}{}^l - \bar{\nabla}_l \bar{R}_{0ij}{}^l + V_{ij} - \epsilon H(|A|^2 + \bar{R}ic(\nu,\nu))g_{ij}$$

$$-\epsilon V g_{ij} + 2n(1-n\epsilon)K_1 \tfrac{|\nabla H|^2}{H^3} g_{ij} + \tfrac{n(1-n\epsilon)K_1}{H}(|A|^2 + \bar{R}ic(\nu,\nu))g_{ij}$$

$$+n(1-n\epsilon)K_1 \tfrac{V}{H^2} g_{ij} + 6n^2 L \tfrac{|\nabla H|^2}{H^4} g_{ij} + \tfrac{2n^2 L}{H^2}(|A|^2 + \bar{R}ic(\nu,\nu))g_{ij}$$

$$+2n^2 L \tfrac{V}{H^3} g_{ij} + 2(H+\lambda)(\epsilon H + n(1-n\epsilon)K_1 \tfrac{1}{H} + \tfrac{n^2 L}{H^2} + kB)h_{ij}$$

The following lemma is crucial for the proof of theorem 1.0.

Lemma 2.11 : *If N satisfies (1.0.2) , and $M \subset N$ is any hypersurface, then*

(*a*) $|\lambda| \leq B$

(*b*) $|\nabla\lambda|^2 \leq 2B^2(|A|^2 + n)$

(*c*) $|V_{ij}| \leq B(2|A| + 1 + c_n \max(K_1, K_2))g_{ij}$

Where $|V_{ij}| \leq c g_{ij}$ means that any eigenvalue l of V_{ij} satisfies the inequality $|l| \leq c$.

Proof: (a) follows directly from lemma 2.7 (a), the first inequality of (1.0.2) and Cauchy-Schwartz. For the proof of (b) we again take normal coordinates around $p \in M$ such that g_{ij} and h_{ij} become diagonal. From lemma 2.7 (b) we get

$$|\nabla\lambda|^2 = \sum_{i=1}^{n} |\lambda_i \bar{g}(e_m, \overrightarrow{H}_{[p]}) + \bar{g}(\nu, \bar{\nabla}_{e_i}\overrightarrow{H}_{[p]})|^2$$

$$\leq 2\sum_{i=1}^{n} (|\lambda_i \bar{g}(e_m, \overrightarrow{H}_{[p]})|^2 + |\bar{g}(\nu, \bar{\nabla}_{e_i}\overrightarrow{H}_{[p]})|^2)$$

and if we use Cauchy-Schwartz and the first two inequalities of (1.0.2) , we obtain

$$|\nabla\lambda|^2 \leq 2\sum_{i=1}^{n} |\lambda_i|^2 B^2 + 2\sum_{i=1}^{n} B^2 = 2B^2(|A|^2 + n)$$

For (c) let in addition u be an eigenvector of V_{ij} , such that $u = e_1$, i.e. $u^i = \delta^i_1$. Then

$$|V_{ij}u^iu^j| = |b_{ij}u^iu^j + \bar{R}(\nu, u, \overrightarrow{H}_{[p]}, u) + 2\sum_{m=1}^{n} h_{1m}a_{m1}|$$

$$\leq 2|\sum_{m=1}^{n} h_{1m}a_{m1}\ | + |b_{ij}u^iu^j| + |\bar{R}(\nu, u, \overrightarrow{H}_{[p]}, u)|$$

Let v be the vector with components $v^j = h_{1j}$. This yields

$$|V_{ij}u^iu^j| \leq 2|a_{ij}v^iu^j| + |b_{ij}u^iu^j| + |\bar{R}(\nu, u, \overrightarrow{H}_{[p]}, u)|$$

and with (1.0.2) finally

$$|V_{ij}u^iu^j| \leq 2|v||u|B + |u|^2B + |\nu||\overrightarrow{H}_{[p]}||u|^2c_n\max(K_1, K_2)$$

$$\leq B(2|v| + 1 + c_n\max(K_1, K_2))$$

which proves (c) since $|v|^2 = h_{1m}h_{1m} \leq h_{mk}h_{mk} = |A|^2$.

The next lemmata are lemmata 2.2, 2.3 in [H2]

Lemma 2.12 : *For anyy* $\eta > 0$ *the following inequalities hold*

(*a*) $|\nabla A|^2 \geq (\frac{3}{n+2} - \eta)|\nabla H|^2 - \frac{2}{n+2}(\frac{2}{n+2}\eta^{-1} - \frac{n}{n-1})|w|^2$

(*b*) $|\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \geq \frac{n-1}{2n+1}|\nabla A|^2 - C(n, K_1, K_2)$

Lemma 2.13 : *If* $H > 0$ *and condition (c) in theorem 1.0 is satisfied with* $\epsilon > 0$*, then*

(*a*) $Z \geq n\epsilon^2H^2(|A|^2 - \frac{1}{n}H^2)$

(*b*) $Q^2 \geq \frac{1}{4}\epsilon^2H^2|\nabla H|^2 - (\frac{c_n\max(K_1,K_2)}{\epsilon})^2H^2$

It is easy to see that the following lemma holds

Lemma 2.14 : $\frac{d}{dt}\lambda = (H+\lambda)\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu) - < \overrightarrow{H}^{0}_{l}, \nabla_l (H+\lambda) >$

We define $\widetilde{H} := H + \lambda$ and obtain from 2.5 and 2.14

Lemma 2.15 : $\frac{d}{dt}\widetilde{H} = \Delta\widetilde{H} - < \nabla_i\widetilde{H}, \overrightarrow{H}^{0}_{i} > + \widetilde{H}(|A|^2 + \bar{R}ic(\nu,\nu)) + \widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu)$

Lemma 2.16 :
$$\begin{aligned}\frac{d}{dt}|\nabla\widetilde{H}|^2 &= \Delta|\nabla\widetilde{H}|^2 - 2|\nabla^2\widetilde{H}|^2 - < \nabla_i|\nabla\widetilde{H}|^2, \overrightarrow{H}^{0}_{i} > \\ &+ 2 < \nabla_i\widetilde{H}h_{jk}, \nabla_j\widetilde{H}h_{ik} > -2 < \bar{R}_{ij}, \nabla_i\widetilde{H}\nabla_j\widetilde{H} > \\ &+ 2|A|^2|\nabla\widetilde{H}|^2 + 2\widetilde{H} < \nabla_i\widetilde{H}, \nabla_i|A|^2 > + 2\bar{R}ic(\nu,\nu)|\nabla\widetilde{H}|^2 \\ &+ 2\widetilde{H} < \nabla_i\widetilde{H}, \bar{\nabla}_i\bar{R}_{0l0}{}^{l} > + 4\widetilde{H} < \nabla_i\widetilde{H}, \bar{R}_{ml0}{}^{l}h^{m}{}_{i} > \\ &- 2g^{ij}\nabla_i\widetilde{H} < \nabla\widetilde{H}, (\bar{\nabla}_{e_j}\overrightarrow{H}_{[p]})^{\perp} > + 2\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu)|\nabla\widetilde{H}|^2 \\ &+ 2\widetilde{H}g^{ij}\nabla_i\widetilde{H}\bar{g}(\bar{\nabla}_{e_j}\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu) + 2\widetilde{H}g^{ij}\nabla_i\widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \frac{\partial F}{\partial x_k})h^{k}{}_{j}\end{aligned}$$

Proof:

$$\begin{aligned}\frac{d}{dt}|\nabla\widetilde{H}|^2 &= 2\widetilde{H} < h_{ij}, \nabla_i\widetilde{H}\nabla_j\widetilde{H} > + 2g^{ij}\nabla_i\widetilde{H}\nabla_j(\Delta\widetilde{H} \\ &- < \nabla_l\widetilde{H}, \overrightarrow{H}^{0}_{l} > + \widetilde{H}(|A|^2 + \bar{R}ic(\nu,\nu)) + \widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu))\end{aligned}$$

We use the relations

$$\begin{aligned}\nabla_j\Delta\widetilde{H} &= \Delta\nabla_j\widetilde{H} - g^{lk}\nabla_l\widetilde{H}(Hh_{jk} - h_{jm}g^{mn}h_{nk} + \bar{R}_{jk}) \\ \nabla_j\bar{R}ic(\nu,\nu) &= \bar{\nabla}_j\bar{R}_{0l0}{}^{l} + 2\bar{R}_{ml0}{}^{l}h^{m}{}_{j} \\ \nabla_j(\widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu)) &= \bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu)\nabla_j\widetilde{H} + \widetilde{H}\bar{g}(\bar{\nabla}_{e_j}\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]},\nu) \\ &+ \widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \frac{\partial F}{\partial x_k})h^{k}{}_{j} \\ \nabla_j\overrightarrow{H}^{0} &= (\bar{\nabla}_{e_j}\overrightarrow{H}_{[p]})^{\perp} + \lambda(\bar{\nabla}_{e_j}e_0)^{\perp}\end{aligned}$$

and get

$$
\begin{aligned}
\frac{d}{dt}|\nabla\widetilde{H}|^2 &= 2\widetilde{H} < h_{ij}, \nabla_i\widetilde{H}\nabla_j\widetilde{H} > \\
&+ 2g^{ij}\nabla_i\widetilde{H}(\Delta\nabla_j\widetilde{H} - g^{kl}\nabla_l\widetilde{H}(Hh_{jk} - h_{jm}g^{mn}h_{nk} + \bar{R}_{jk})) \\
&+ 2 < \nabla_i\widetilde{H}, \nabla_i(\widetilde{H}|A|^2) > + 2\widetilde{H}g^{ij}\nabla_i\widetilde{H}(\bar{\nabla}_j\bar{R}_{0l0}{}^{l} + 2\bar{R}_{ml0}{}^{l}h^{m}{}_{j}) \\
&+ 2\bar{R}ic(\nu,\nu)|\nabla\widetilde{H}|^2 - 2g^{ij}\nabla_i\widetilde{H} < \nabla\widetilde{H}, \nabla_j\overrightarrow{H}^0 > \\
&- < \nabla_i|\nabla\widetilde{H}|^2, \overrightarrow{H}^0_i > + 2g^{ij}\nabla_i\widetilde{H}(\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \nu)\nabla_j\widetilde{H} \\
&+ \widetilde{H}\bar{g}(\bar{\nabla}_{e_j}\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \nu) + \widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \frac{\partial F}{\partial x_k})h^{k}{}_{j}) \\
&= \Delta|\nabla\widetilde{H}|^2 - 2|\nabla^2\widetilde{H}|^2 - < \nabla_i|\nabla\widetilde{H}|^2, \overrightarrow{H}^0_i > \\
&+ 2\lambda < h_{ij}, \nabla_i\widetilde{H}\nabla_j\widetilde{H} > \\
&+ 2 < \nabla_i\widetilde{H}h_{jk}, \nabla_j\widetilde{H}h_{ik} > - 2 < \bar{R}_{ij}, \nabla_i\widetilde{H}\nabla_j\widetilde{H} > \\
&+ 2|A|^2|\nabla\widetilde{H}|^2 + 2\widetilde{H} < \nabla_i\widetilde{H}, \nabla_i|A|^2 > + 2\bar{R}ic(\nu,\nu)|\nabla\widetilde{H}|^2 \\
&+ 2\widetilde{H} < \nabla_i\widetilde{H}, \bar{\nabla}_i\bar{R}_{0l0}{}^{l} > + 4\widetilde{H} < \nabla_i\widetilde{H}, \bar{R}_{ml0}{}^{l}h^{m}{}_{i} > \\
&- 2g^{ij}\nabla_i\widetilde{H} < \nabla\widetilde{H}, (\bar{\nabla}_{e_j}\overrightarrow{H}_{[p]})^{\perp} > \\
&- 2\lambda g^{ij}\nabla_i\widetilde{H}g^{kl}\nabla_k\widetilde{H} < e_l, (\bar{\nabla}_{e_j}e_0)^{\perp} > + 2\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \nu)|\nabla\widetilde{H}|^2 \\
&+ 2\widetilde{H}g^{ij}\nabla_i\widetilde{H}\bar{g}(\bar{\nabla}_{e_j}\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \nu) + 2\widetilde{H}g^{ij}\nabla_i\widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \frac{\partial F}{\partial x_k})h^{k}{}_{j}
\end{aligned}
$$

and the lemma follows from the equation $h_{lj} = < e_l, (\bar{\nabla}_{e_j}e_0)^{\perp} >$.

3. Convexity and pinching:

__**Lemma 3.0 :**__ *Assume that* $H > n\sqrt{K_1} + lB$ *on* M_0 *with a constant* l *as in theorem 1.0. Then this is true on* M_t *for each* $t \in [0, T)$

Proof: We can find a positive constant η such that

$$H > n\sqrt{K_1} + lB + \eta$$

Then we look at the first time t_0 where at some point on M_{t_0}:

$$H = n\sqrt{K_1} + lB + \eta$$

From $|A|^2 \geq \frac{H^2}{n}$ and $\bar{R}ic(\nu,\nu) \geq -nK_1$ and together with lemma 2.11 (c) we obtain

$$H(|A|^2 + \bar{R}ic(\nu,\nu)) + V \geq \frac{H^2}{\sqrt{n}}|A| - nK_1H - nB(2|A| + 1 + c_n \max(K_1, K_2))$$

$$\geq (\frac{n^2K_1 + l^2B^2 + \eta^2}{\sqrt{n}} - 2nB)|A| - nK_1H - nB(1 + c_n \max(K_1, K_2))$$

$$\geq (\frac{l^2B^2 + \eta^2}{\sqrt{n}} - 2nB)|A| - nB(1 + c_n \max(K_1, K_2))$$

and since the condition for l implies that $(\frac{l^2B}{\sqrt{n}} - 2n)B \geq 0$, we conclude that this is not smaller than

$$(\frac{l^2B}{\sqrt{n}} - 2n)B\frac{H}{\sqrt{n}} - nB(1 + c_n \max(K_1, K_2)) + \frac{\eta^2}{\sqrt{n}}|A|$$

$$\geq B\left((\frac{l^2B}{n} - 2\sqrt{n})lB - n(1 + c_n \max(K_1, K_2))\right) + \frac{\eta^3}{n}$$

$$\geq \frac{\eta^3}{n} > 0$$

and the proof follows from the parabolic maximum principle and 2.8 (d).

Our aim is to prove:

Theorem 3.1 : *If N satisfies* (1.0.2) *and if $M_{ij} \geq 0$ on M_0 with a constant k as in theorem 1.0, then $M_{ij} \geq 0$ for all $t \in [0,T)$.*

For the proof we use the maximum principle for tensors due to Hamilton [Ha]:

Theorem 3.2 : *Let $\{u^k\}$ be a vektor field and let g_{ij}, M_{ij} and N_{ij} symmetric tensors on M_t , which may all depend smoothly on time. Assume N_{ij} satisfies a null-eigenvector condition, i.e. for every null-eigenvector $\{v^i\}$ of M_{ij} we have $N_{ij}v^iv^j \geq 0$. Assume in addition that M_{ij} is a solution of the evolution equation*

$$\frac{d}{dt}M_{ij} = \Delta M_{ij} + < u_k, \nabla_k M_{ij} > + N_{ij}$$

and that $M_{ij} \geq 0$ on M_0 .

Then this is true on M_t for $t \in [0,T)$.

Proof of 3.1: Here we have

$$\begin{aligned} N_{ij} = & -2(H+\lambda)h_{im}g^{mn}h_{nj} + (|A|^2 + \bar{R}ic(\nu,\nu))h_{ij} \\ & - h_{jl}\bar{R}^l{}_{mi}{}^m - h_{il}\bar{R}^l{}_{mj}{}^m + 2h_{lm}\bar{R}^l{}_i{}^m{}_j \\ & - \bar{\nabla}_j\bar{R}_{0li}{}^l - \bar{\nabla}_l\bar{R}_{0ij}{}^l + V_{ij} - \epsilon H(|A|^2 + \bar{R}ic(\nu,\nu))g_{ij} - \epsilon V g_{ij} \\ & + 2n(1-n\epsilon)K_1\frac{|\nabla H|^2}{H^3}g_{ij} + \frac{n(1-n\epsilon)K_1}{H}(|A|^2 + \bar{R}ic(\nu,\nu))g_{ij} \\ & + n(1-n\epsilon)K_1\frac{V}{H^2}g_{ij} + 6n^2L\frac{|\nabla H|^2}{H^4}g_{ij} + \frac{2n^2L}{H^2}(|A|^2 + \bar{R}ic(\nu,\nu))g_{ij} \\ & + 2n^2L\frac{V}{H^3}g_{ij} + 2(H+\lambda)(\epsilon H + n(1-n\epsilon)K_1\frac{1}{H} \\ & + \frac{n^2L}{H^2} + kB)h_{ij} \end{aligned}$$

Note that by lemma 2.7 (c) V_{ij} is a symmetric tensor, so we must only prove that N_{ij} satisfies the null-eigenvector condition. For this purpose we look at the first time t_0, where at a point $p \in M_{t_0}$ a null-eigenvector $v = \{v^i\}$ of M_{ij} occurs and choose orthonormal vectors $(e_1, \ldots, e_n)$ for $T_pM_{t_0}$, such that h_{ij} and M_{ij} become diagonal. We also choose $e_1 = v$ and from $M_{11} = 0$, $M_{ii} \geq 0$ in p it follows:

$$\lambda_1 = \epsilon H + n(1 - n\epsilon)K_1 \frac{1}{H} + \frac{n^2 L}{H^2} + kB$$

$$\lambda_i \geq \epsilon H + n(1 - n\epsilon)K_1 \frac{1}{H} + \frac{n^2 L}{H^2} + kB$$

$$H \geq nkB$$

$$\lambda_i \geq \lambda_1$$

From this we obtain

$$\begin{aligned}
N_{ij}v^iv^j &= N_{11} \\
&= -2(H+\lambda)\lambda_1^2 + (|A|^2 + \bar{R}ic(\nu,\nu))\lambda_1 + 2\sum_{m=1}^{n}(\lambda_m - \lambda_1)\bar{R}_{1m1m} \\
&- \sum_{m=1}^{n}(\bar{\nabla}_1\bar{R}_{0m1m} + \bar{\nabla}_m\bar{R}_{011m}) + V_{11} - \epsilon H(|A|^2 + \bar{R}ic(\nu,\nu)) - \epsilon V \\
&+ 2n(1-n\epsilon)K_1\frac{|\nabla H|^2}{H^3} + \frac{n(1-n\epsilon)K_1}{H}(|A|^2 + \bar{R}ic(\nu,\nu)) \\
&+ n(1-n\epsilon)K_1\frac{V}{H^2} + 6n^2L\frac{|\nabla H|^2}{H^4} + \frac{2n^2L}{H^2}(|A|^2 + \bar{R}ic(\nu,\nu)) \\
&+ 2n^2L\frac{V}{H^3} + 2(H+\lambda)(\epsilon H + n(1-n\epsilon)K_1\frac{1}{H} + \frac{n^2L}{H^2} + kB)\lambda_1
\end{aligned}$$

$$
\begin{aligned}
&\geq (|A|^2 + \bar{R}ic(\nu,\nu))\lambda_1 - 2K_1(H - n\lambda_1) - 2nL + V_{11} - \epsilon H(|A|^2 + \bar{R}ic(\nu,\nu)) \\
&- \epsilon V + \frac{n(1-n\epsilon)K_1}{H}(|A|^2 + \bar{R}ic(\nu,\nu)) + n(1-n\epsilon)K_1\frac{V}{H^2} \\
&+ \frac{2n^2L}{H^2}(|A|^2 + \bar{R}ic(\nu,\nu)) + 2n^2L\frac{V}{H^3}
\end{aligned}
$$

$$
\begin{aligned}
&= (|A|^2 + \bar{R}ic(\nu,\nu))(\frac{2n(1-n\epsilon)K_1}{H} + \frac{3n^2L}{H^2} + kB) - 2K_1H \\
&+ 2nK_1(\epsilon H + \frac{n(1-n\epsilon)K_1}{H} + \frac{n^2L}{H^2} + kB) - 2nL \\
&+ V_{11} - \epsilon V + n(1-n\epsilon)K_1\frac{V}{H^2} + 2n^2L\frac{V}{H^3}
\end{aligned}
$$

Taking into account that $|A|^2 \geq \frac{H^2}{n}$ and $\bar{R}ic(\nu,\nu) \geq -nK_1$, it follows

$$
\begin{aligned}
N_{11} &\geq 2(1-n\epsilon)K_1H + 3nL + kB|A|^2 - \frac{2n^2K_1^2(1-n\epsilon)}{H} - \frac{3n^3LK_1}{H^2} - nK_1kB \\
&- 2K_1H + 2nK_1\epsilon H + 2\frac{n^2K_1^2(1-n\epsilon)}{H} + \frac{2n^3LK_1}{H^2} + 2nK_1kB \\
&- 2nL + V_{11} - \epsilon V + n(1-n\epsilon)\frac{K_1V}{H^2} + 2n^2L\frac{V}{H^3} \\
&= \frac{nL}{H^2}(H^2 - n^2K_1) + kB|A|^2 + nK_1kB + V_{11} - \epsilon V + n(1-n\epsilon)\frac{K_1V}{H^2} + 2n^2L\frac{V}{H^3}
\end{aligned}
$$

and together with lemma 2.11 (c)

$$
\begin{aligned}
N_{11} &\geq kB|A|^2 + kBnK_1 - (1+n\epsilon)B(2|A| + 1 + c_n\max(K_1,K_2)) \\
&- n^2B(\frac{(1-n\epsilon)K_1}{H^2} + 2\frac{nL}{H^3})(2|A| + 1 + c_n\max(K_1,K_2))
\end{aligned}
$$

Since $|A|^2 \leq H^2$ and $\epsilon < \frac{1}{n}$, we get

$$
\begin{aligned}
N_{11} &\geq kB\frac{H^2}{n} - 2B(2H + 1 + c_n\max(K_1,K_2)) \\
&- n^2B(\frac{K_1}{H^2} + 2\frac{nL}{H^3})(2H + 1 + c_n\max(K_1,K_2))
\end{aligned}
$$

and the theorem is proved if $B = 0$. In the case where $B > 0$ we conclude

$$
\begin{aligned}
N_{11} &\geq HB(\frac{kH}{n} - 4) - 2B(1 + c_n\max(K_1,K_2)) \\
&- n^2B(K_1 + \frac{2L}{kB})(\frac{2}{H} + \frac{1 + c_n\max(K_1,K_2)}{H^2})
\end{aligned}
$$

and since the condition for k implies that $k^2B-4>0$, we can continue with

$$\begin{aligned}N_{11} &\geq nkB^2(k^2B-4)-2B(1+c_n\max(K_1,K_2))\\&\quad -n^2B(K_1+2\frac{L}{kB})(\frac{2}{nkB}+\frac{1+c_n\max(K_1,K_2)}{n^2k^2B^2})\\&\geq 0\end{aligned}$$

Then the next corollary is immediate

Corollary 3.3 : *There are constants* $d_1,\ d_2,\ d_3>0$ *, such that for all* $t\in[0,T)$*:*

$$d_1<H<d_2\widetilde{H}<d_3H$$

Corollary 3.4 : $T<\infty$

Proof: Lemmata 3.0 and 2.11 (c) imply the existence of a constant $\eta>0$, with

$$\frac{d}{dt}H\geq \Delta H-<\nabla_iH,\overrightarrow{H}_i^0>+\eta H^3$$

and therefore

$$H\geq\frac{H_{min}(0)}{\sqrt{1-2\eta H_{min}^2(0)t}}$$

This gives $T<\infty$.

In the next lemma we investigate expressions on $\widetilde{N}$ and $\widetilde{M_t}$ which we denote by a tilde.

Lemma 3.5 : *Suppose that* $\widetilde{H}>(n+k)\sqrt{\widetilde{K}_1}$ *holds on* $\widetilde{M_0}$*. Then we can find a constant* c *, such that* $|\widetilde{A}|^2<c\widetilde{H}^2$ *, for* $t\in[0,T)$ *.*

Proof: By [H2] corollary 3.5 and with $m:=n+k$ we have

$$\begin{aligned}\frac{d}{dt}(|\widetilde{A}|^2-c\widetilde{H}^2)&=\widetilde{\Delta}(|\widetilde{A}|^2-c\widetilde{H}^2)-\frac{2\widetilde{Q}^2}{\widetilde{H}^2}-\frac{2}{\widetilde{H}}<\widetilde{\nabla}_i(|\widetilde{A}|^2-c\widetilde{H}^2),\widetilde{\nabla}_i\widetilde{H}>\\&\quad+\frac{2}{\widetilde{H}^2}(|\widetilde{A}|^2-c\widetilde{H}^2)|\widetilde{\nabla}\widetilde{H}|^2+2(|\widetilde{A}|^2-c\widetilde{H}^2)(|\widetilde{A}|^2+\widetilde{R}ic(\widetilde{\nu},\widetilde{\nu}))\\&\quad-4(\widetilde{h}^{ij}\widetilde{h}_{jm}\widetilde{R}_{mli}{}^{l}-\widetilde{h}^{ij}\widetilde{h}^{lm}\widetilde{R}_{milj})-2\widetilde{h}^{ij}(\widetilde{\nabla}_j\widetilde{R}_{0li}{}^{\ l}+\widetilde{\nabla}_l\widetilde{R}_{0ij}{}^{\ l})\end{aligned}$$

and by (1.0.1)

$$-2\widetilde{h}^{ij}(\widetilde{\nabla}_j\widetilde{R}_{0li}{}^{\;l} + \widetilde{\nabla}_l\widetilde{R}_{0ij}{}^{\;l}) \le |\widetilde{A}|^2 + c_1$$

with a constant c_1 depending only on $\widetilde{N}$. Furtheron

$$-4(\widetilde{h}^{ij}\widetilde{h}_{jm}\widetilde{R}_{mli}{}^{l} - \widetilde{h}^{ij}\widetilde{h}^{lm}\widetilde{R}_{milj}) \le 4m\widetilde{K}_1(|\widetilde{A}|^2 - \frac{1}{m}\widetilde{H}^2)$$

and we conclude

$$\begin{aligned}\frac{d}{dt}(|\widetilde{A}|^2 - c\widetilde{H}^2) \le \widetilde{\Delta}(|\widetilde{A}|^2 - c\widetilde{H}^2) - \frac{2\widetilde{Q}^2}{\widetilde{H}^2} - \frac{2}{\widetilde{H}} < \widetilde{\nabla}_i(|\widetilde{A}|^2 - c\widetilde{H}^2), \widetilde{\nabla}_i\widetilde{H} > \\ + \frac{2}{\widetilde{H}^2}(|\widetilde{A}|^2 - c\widetilde{H}^2)|\widetilde{\nabla}\widetilde{H}|^2 + 2(|\widetilde{A}|^2 - c\widetilde{H}^2)(|\widetilde{A}|^2 + \widetilde{R}ic(\widetilde{\nu},\widetilde{\nu})) \\ + |\widetilde{A}|^2 + c_1 + 4m\widetilde{K}_1(|\widetilde{A}|^2 - \frac{1}{m}\widetilde{H}^2)\end{aligned}$$

By [H2] section 4 we have $\widetilde{H} > m\sqrt{\widetilde{K}_1}$ for $t \in [0,T)$ and that there exists a positive constant η with $\widetilde{H}^2 > \eta$ for $t \in [0,T)$. Now let

$$\sigma := \max(4m\widetilde{K}_1 \ , \ 4)$$

$$c > \max(\frac{2\sigma^2 + c_1 - \sigma}{\eta} \ , \ \frac{2}{m})$$

and c so large that

$$|\widetilde{A}|^2 - c\widetilde{H}^2 < -\sigma \tag{3.0.0}$$

on $\widetilde{M_0}$. Then (3.0.0) remains true for $t \in [0,T)$. For if t_0 is the first time where (3.0.0) fails in a point $p \in \widetilde{M_{t_0}}$ then in this point

$$\begin{aligned}0 \le \frac{d}{dt}(|\widetilde{A}|^2 - c\widetilde{H}^2) &\le -2\sigma(|\widetilde{A}|^2 + \widetilde{R}ic(\widetilde{\nu},\widetilde{\nu})) + 4m\widetilde{K}_1(|\widetilde{A}|^2 - \frac{1}{m}\widetilde{H}^2) + |\widetilde{A}|^2 + c_1 \\ &\le -2\sigma(c\widetilde{H}^2 - \sigma + \widetilde{R}ic(\widetilde{\nu},\widetilde{\nu})) + 4m\widetilde{K}_1(c - \frac{1}{m})\widetilde{H}^2 + c\widetilde{H}^2 - \sigma + c_1 \\ &\le -2\sigma\left((c - \frac{1}{m})\widetilde{H}^2 - \sigma\right) + \left(4m\widetilde{K}_1(c - \frac{1}{m}) + c\right)\widetilde{H}^2 + c_1 - \sigma \\ &= \left((4m\widetilde{K}_1 - 2\sigma)(c - \frac{1}{m}) + c\right)\widetilde{H}^2 + 2\sigma^2 + c_1 - \sigma\end{aligned}$$

The assumptions for c and σ imply

$$\begin{aligned}(4m\widetilde{K}_1 - 2\sigma)(c - \frac{1}{m}) + c &\leq -\sigma(c - \frac{1}{m}) + c \\ &\leq -\sigma\frac{c}{2} + c \leq -c\end{aligned}$$

and therefore

$$\left((4m\widetilde{K}_1 - 2\sigma)(c - \frac{1}{m}) + c\right)\widetilde{H}^2 + 2\sigma^2 + c_1 - \sigma$$

$$\leq -c\eta + 2\sigma^2 + c_1 - \sigma < 0$$

This contradiction proves lemma 3.5.

4. The eigenvalues λ_i of the second fundamental form approach each other

We want to show that the eigenvalues of the second fundamental form asymptotically approach each other in those points on M_t where the mean curvature becomes large. As in [H2] we investigate the function

$$f_\sigma := \frac{|A|^2 - \frac{1}{n}H^2}{H^{2-\sigma}}$$

and show

Theorem 4.0 : *There exist constants c, σ , depending only on M_0 such that*

$$f_\sigma \le c$$

for all $t \in [0,T)$.

For the proof we need the evolution equation for f_σ

Lemma 4.1 :

$$\begin{aligned}\frac{d}{dt}f_\sigma =& \Delta f_\sigma - < \nabla_l f_\sigma, \overrightarrow{H}^0_l > +2(1-\sigma)\frac{1}{H} < \nabla_l f_\sigma, \nabla_l H > \\ &- 2\frac{Q^2}{H^{4-\sigma}} - \sigma(1-\sigma) f_\sigma \frac{|\nabla H|^2}{H^2} + \sigma f_\sigma(|A|^2 + \bar{R}ic(\nu,\nu)) \\ &- \frac{1}{H^{2-\sigma}}(4(h^{ij}h_j^{\ m}\bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}) + 2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^l - \bar{\nabla}_l \bar{R}_{0ij}{}^l)) \\ &+ \frac{2}{H^{2-\sigma}} < h^0_{ij}, V_{ij} > -(2-\sigma) f_\sigma \frac{V}{H}\end{aligned}$$

where $h^0_{ij} := h_{ij} - \frac{1}{n}Hg_{ij}$ is the tracefree second fundamental form.

Proof: This is a direct consequence of the evolution equations derived in section 2.

Lemma 4.2 : *There is a positive constant c depending only on M_0 such that for all $\sigma < 1$*

$$\begin{aligned}\frac{d}{dt}f_\sigma \le& \Delta f_\sigma - < \nabla_i f_\sigma, \overrightarrow{H}^0_i > +2(1-\sigma)\frac{1}{H} < \nabla_i f_\sigma, \nabla_i H > \\ &- \frac{\epsilon^2}{2}\frac{|\nabla H|^2}{H^{2-\sigma}} + \sigma f_\sigma |A|^2 + c f_\sigma + cH^\sigma\end{aligned}$$

Proof: We have

$$h^{ij}h_j^{\ m}\bar{R}_{mli}{}^{l} - h^{ij}h^{lm}\bar{R}_{milj} = \sum_{l<m}(\lambda_l - \lambda_m)^2\bar{R}_{lmlm}$$
$$\geq -K_1\sum_{l<m}(\lambda_l - \lambda_m)^2 = -nK_1(|A|^2 - \frac{1}{n}H^2)$$

and

$$h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^{l} - \bar{\nabla}_l\bar{R}_{0ij}{}^{l}) =< h_{ij}, \bar{\nabla}_j\bar{R}_{0li}{}^{l} - \bar{\nabla}_l\bar{R}_{0ij}{}^{l} >=< h^0_{ij}, \bar{\nabla}_j\bar{R}_{0li}{}^{l} - \bar{\nabla}_l\bar{R}_{0ij}{}^{l} >$$

$$|h^0_{ij}|^2 = |A|^2 - \frac{1}{n}H^2$$

and consequently

$$\begin{aligned}
&-\frac{1}{H^{2-\sigma}}(4(h^{ij}h_j^{\ m}\bar{R}_{mli}{}^{l} - h^{ij}h^{lm}\bar{R}_{milj}) + 2h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^{l} - \bar{\nabla}_l\bar{R}_{0ij}{}^{l}))\\
&\leq \frac{1}{H^{2-\sigma}}\left(4nK_1(|A|^2 - \frac{1}{n}H^2) + 2(\frac{|A|^2 - \frac{1}{n}H^2}{2} + \frac{1}{2}|\bar{\nabla}_j\bar{R}_{0li}{}^{l} - \bar{\nabla}_l\bar{R}_{0ij}{}^{l}|^2)\right)\\
&\leq cf_\sigma + \frac{c}{H^{2-\sigma}}
\end{aligned}$$

with a positive constant c depending only on M_0. Then by lemma 3.0, theorem 3.1 and lemma 2.11 (c)

$$|V_{ij}|^2 \leq cH^2$$

$$|V| \leq cH$$

and then

$$\frac{2}{H^{2-\sigma}} < h^0_{ij}, V_{ij} > -(2-\sigma)f_\sigma\frac{V}{H} \leq cf_\sigma + cH^\sigma$$

Finally with lemma 2.13

$$\begin{aligned}
\frac{d}{dt}f_\sigma \leq \Delta f_\sigma - &< \nabla_i f_\sigma, \overrightarrow{H}^0_i > + 2(1-\sigma)\frac{1}{H} < \nabla_i f_\sigma, \nabla_i H > - \frac{\epsilon^2}{2}\frac{|\nabla H|^2}{H^{2-\sigma}}\\
&+ \sigma f_\sigma|A|^2 + cf_\sigma + \frac{c}{H^{2-\sigma}} + cH^\sigma
\end{aligned}$$

This proves 4.2 since $H > c(M_0) > 0$.

Exactly as in [H2] we have

Lemma 4.3 *There is a positive constant* c_0 *depending on* $M_0,\ \epsilon,\ K_1,\ K_2,\ L$ *such that for all* $p \geq 2,\ 0 \leq \sigma \leq \frac{1}{2}$ *and* $\eta > 0$

$$\tfrac{1}{2} n\epsilon^2 \int f_\sigma^p H^2 d\mu \leq (2\eta p + 5) \int \tfrac{f^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu + \tfrac{p-1}{\eta} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + c_o^p$$

Since we cannot show by the maximum principle that f_σ is bounded for suitable $\sigma > 0$ we want to show that high L^p -norms of f_σ are bounded. First we conclude from lemma 4.2 and the inequalities $\widetilde{H} > 0$, $H > c(M_0) > 0$ that we can find a positive constant c_1 depending only on $M_0,\ \epsilon,\ K_1,\ K_2,\ L$ such that

$$\begin{aligned} \text{(4.0.0)} \qquad \frac{d}{dt} \int f_\sigma^p d\mu \leq & -p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu - p \int f_\sigma^{p-1} < \nabla_i f_\sigma, \overrightarrow{H}_i^0 > d\mu \\ & + 2p(1-\sigma) \int \frac{f_\sigma^{p-1}}{H} |\nabla H| |\nabla f_\sigma| d\mu - \frac{\epsilon^2 p}{2} \int \frac{f_\sigma^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu \\ & + p\sigma \int f_\sigma^p H^2 d\mu + c_1 p \int f_\sigma^p d\mu + c_1 p \int f_\sigma^{p-1} H^\sigma d\mu \end{aligned}$$

Furtheron we have (see [H2])

(4.0.1)

$$2p(1-\sigma) \int \frac{f_\sigma^{p-1}}{H} |\nabla f_\sigma| |\nabla H| d\mu \leq \frac{p(p-1)}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + \frac{2p}{p-1} \int \frac{f_\sigma^{p-1}}{H^{2-\sigma}} |\nabla H|^2 d\mu$$

and since $|\overrightarrow{H}^0|^2 \leq \bar{g}(\overrightarrow{H}_{[p]}, \overrightarrow{H}_{[p]}) \leq B^2$ we obtain by Young's inequality for arbitrary $\eta_1 > 0$

$$-p \int f_\sigma^{p-1} < \nabla_i f_\sigma, \overrightarrow{H}_i^0 > d\mu \ \leq \ \frac{pB}{2\eta_1} \int f_\sigma^p d\mu + \frac{\eta_1 p B}{2} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu$$

If we choose $\eta_1 = \frac{p-1}{2B}$, if $B \neq 0$ and $p \geq 2$ we conclude

$$\text{(4.0.2)} \qquad -p \int f_\sigma^{p-1} < \nabla_i f_\sigma, \overrightarrow{H}_i^0 > d\mu \ \leq \ \frac{p(p-1)}{4} \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + 2B^2 \int f_\sigma^p d\mu$$

If $B = 0$, (4.0.2) is trivial.

Now we estimate

$$c_1p\int f_\sigma^{p-1}H^\sigma d\mu = c_1p\int f_\sigma^{p-2}\cdot Hf_\sigma\cdot H^{\sigma-1}d\mu \le c_1p\int f_\sigma^{p-2}(\frac{H^2f_\sigma^2}{2\eta}+\frac{\eta}{2}H^{2\sigma-2})d\mu$$

and with $\eta=\frac{16c_1\sqrt{p}}{n\epsilon^3}$, the relation $x^{p-2}\le x^p+1$, for each $x\ge 0$ and the fact that $|M_t|$ decreases in time and $\frac{1}{H^{2-2\sigma}}\le const(M_0)$ for $\sigma\le 1$, we obtain for $\sigma\le 1,\ p\ge 2$

$$c_1p\int f_\sigma^{p-1}H^\sigma d\mu \ \le\ \frac{n\epsilon^3\sqrt{p}}{32}\int f_\sigma^pH^2d\mu + c_2p^2\int f_\sigma^p d\mu + c_2p^2 \tag{4.0.3}$$

In a first step the relations (4.0.0), (4.0.1), (4.0.2) and (4.0.3) give

$$\begin{aligned}\frac{d}{dt}\int f_\sigma^p d\mu \le & -\frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2d\mu + 2\frac{p}{p-1}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2d\mu\\ & -\frac{\epsilon^2p}{2}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2d\mu + c_3p^2\int f_\sigma^pd\mu + c_3p^2\\ & + p\sigma\int f_\sigma^pH^2d\mu + \frac{n\epsilon^3\sqrt{p}}{32}\int f_\sigma^pH^2d\mu\end{aligned} \tag{4.0.4}$$

<u>Lemma 4.4</u> : *There is a positive constant* $c(M_0,n,\epsilon,K_1,K_2,L,B)$ *, such that for all*

$p\ge\max(\frac{25}{\epsilon^2},\ 2),\ \sigma\le\min(\frac{1}{2},\ \frac{n\epsilon^3}{32\sqrt{p}})$

$$(\int f_\sigma^pd\mu)^{\frac{1}{p}}\le c$$

Proof:

Since $\frac{2p}{p-1}\le\frac{\epsilon^2p}{4}$ and $p\sigma\le\frac{n\epsilon^3\sqrt{p}}{32}$, we get with (4.0.4)

$$\begin{aligned}\frac{d}{dt}\int f_\sigma^pd\mu \le & -\frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2d\mu - \frac{\epsilon^2p}{4}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2d\mu\\ & + \frac{n\epsilon^3\sqrt{p}}{16}\int f_\sigma^pH^2d\mu + c_3p^2\int f_\sigma^pd\mu + c_3p^2\end{aligned}$$

Now using lemma 4.3 with $\eta = \frac{\epsilon}{2\sqrt{p}}$, we obtain

$$\frac{d}{dt}\int f_\sigma^p d\mu \le -\frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu - \frac{\epsilon^2 p}{4}\int \frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$+\frac{\epsilon\sqrt{p}}{8}\left((\epsilon\sqrt{p}+5)\int \frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu + \frac{2(p-1)\sqrt{p}}{\epsilon}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu + c_0^p\right)$$
$$+c_3 p^2 \int f_\sigma^p d\mu + c_3 p^2$$

and since $5 \le \epsilon\sqrt{p}$, finally

$$\frac{d}{dt}\int f_\sigma^p d\mu \le p^2 c_4^p \int f_\sigma^p d\mu + p^2 c_4^p$$

and therefore

$$\int_{M_t} f_\sigma^p d\mu \le (\int_{M_0} f_\sigma^p d\mu + 1)e^{p^2 c_4^p t} - 1$$

and the result follows from $T < \infty$.

<u>Lemma 4.5 :</u> *Assume* $p \ge \max((\frac{64m}{\epsilon^3 n})^2,\ 2)$, $\sigma \le \min(\frac{1}{2} - \frac{\epsilon^3 n}{64\sqrt{p}},\ \frac{\epsilon^3 n}{64\sqrt{p}})$. *Then*

$$(\int H^m f_\sigma^p d\mu)^{\frac{1}{p}} \le c$$

Proof:

This follows directly from lemma 4.4 since

$$\int H^m f_\sigma^p d\mu = \int f_{\sigma'}^p d\mu$$

with $\sigma' = \sigma + \frac{m}{p} \le \min(\frac{1}{2},\ \frac{n\epsilon^3}{32\sqrt{p}})$

If we define the function $f_{\sigma,\ k}$ by $f_{\sigma,\ k} := \max(f_\sigma - k,\ 0)$ with $k \ge k_0 = \max_{M_0} f_\sigma$, and if we use the results above we obtain as usual (see e.g. [H1] and [H2]) that $f_{\sigma,\ k} \equiv 0$ with a suitable k. This proves that f_σ is bounded.

5. An estimate for $|\nabla H|^2$

We want to prove

Theorem 5.0 : *For any* $\eta > 0$ *there is a constant* $c = c(\eta,\ M_0,\ n)$ *, such that*

$$|\nabla H|^2 \leq \eta H^4 + c$$

It will turn out that the evolution equation for $|\nabla \widetilde{H}|^2$ is much more useful for the proof of this theorem than the equation for $|\nabla H|^2$.

Since

$$|\nabla \widetilde{H}|^2 = |\nabla H + \nabla \lambda|^2 \leq 2|\nabla H|^2 + 2|\nabla \lambda|^2$$

$$|\nabla H|^2 = |\nabla \widetilde{H} - \nabla \lambda|^2 \leq 2|\nabla \widetilde{H}|^2 + 2|\nabla \lambda|^2$$

we get from lemma 2.11 (b) and lemma 3.0

(5.0.0) $$|\nabla \widetilde{H}|^2 \leq 2|\nabla H|^2 + 4B^2(|A|^2 + n) \leq 2|\nabla H|^2 + c_5 H^2$$

(5.0.1) $$|\nabla H|^2 \leq 2|\nabla \widetilde{H}|^2 + 4B^2(|A|^2 + n) \leq 2|\nabla \widetilde{H}|^2 + c_5 H^2$$

with a positive constant c_5 .

The following equation holds

$$2\widetilde{H} < \nabla_i \widetilde{H}, \nabla_i |A|^2 >= 2\widetilde{H}^2 |\nabla A|^2 + 2|A|^2 |\nabla \widetilde{H}|^2 - 2|\nabla_i \widetilde{H} h_{kl} - \nabla_i h_{kl} \widetilde{H}|^2$$

and corollary 3.3, lemma 2.12 and (5.0.0) imply

(5.0.2) $$2\widetilde{H} < \nabla_i \widetilde{H}, \nabla_i |A|^2 > \leq \ c_6 H^2 |\nabla A|^2 + c_6 H^4$$

The next estimate can be derived from the evolution equation of $|\nabla\widetilde{H}|^2$, lemma 2.11, corollary 3.3, the relations (1.0.2), (5.0.0), (5.0.2) and with Cauchy-Schwartz.

We can find a positive constant c_7 depending only on M_0 with

$$\frac{d}{dt}|\nabla\widetilde{H}|^2 \leq \Delta|\nabla\widetilde{H}|^2 - < \nabla_i|\nabla\widetilde{H}|^2, \overrightarrow{H}^0_i > -2|\nabla^2\widetilde{H}|^2 + c_7H^2|\nabla A|^2 + c_7H^4 \tag{5.0.3}$$

We also need the estimate

Lemma 5.1 : *For a suitable positive constant* c_8*:*

$$\frac{d}{dt}(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}) \leq \Delta(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}) - < \nabla_i(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}), \overrightarrow{H}^0_i > +c_8H|\nabla A|^2 + c_8H^3$$

Proof: First we obtain from (5.0.3) and lemma 2.15 that

$$\begin{aligned}\frac{d}{dt}(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}) \leq &\frac{1}{\widetilde{H}}(\Delta|\nabla\widetilde{H}|^2 - < \nabla_i|\nabla\widetilde{H}|^2, \overrightarrow{H}^0_i > -2|\nabla^2\widetilde{H}|^2 + c_7H^2|\nabla A|^2 + c_7H^4)\\ &- \frac{|\nabla\widetilde{H}|^2}{\widetilde{H}^2}(\Delta\widetilde{H} - < \nabla_i\widetilde{H}, \overrightarrow{H}^0_i > +\widetilde{H}(|A|^2 + \bar{R}ic(\nu,\nu)) + \widetilde{H}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \nu))\end{aligned}$$

Since

$$\Delta(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}) = \frac{\Delta|\nabla\widetilde{H}|^2}{\widetilde{H}} - \frac{|\nabla\widetilde{H}|^2}{\widetilde{H}^2}\Delta\widetilde{H} + 2\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}^3}|\nabla\widetilde{H}|^2 - \frac{4}{\widetilde{H}^3} < \nabla_i\widetilde{H}\nabla_j\widetilde{H}, \widetilde{H}\nabla_i\nabla_j\widetilde{H} >$$

, it follows from Cauchy-Schwartz' inequality that

$$\begin{aligned}\frac{d}{dt}(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}) \leq &\Delta(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}) - < \nabla_i(\frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}), \overrightarrow{H}^0_i > +c_7\frac{H^2}{\widetilde{H}}|\nabla A|^2\\ &+ c_7\frac{H^4}{\widetilde{H}} - \frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}(|A|^2 + \bar{R}ic(\nu,\nu)) - \frac{|\nabla\widetilde{H}|^2}{\widetilde{H}}\bar{g}(\bar{\nabla}_{e_0}\overrightarrow{H}_{[p]}, \nu)\end{aligned}$$

and we continue with corollary 3.3, lemma 2.12 and (1.0.2), (5.0.0)

Lemma 5.2 : *There is a constant* c_{11} *, such that*

$$\frac{d}{dt}\left(H(|A|^2-\frac{1}{n}H^2)\right)\leq\Delta\left(H(|A|^2-\frac{1}{n}H^2)\right)-<\nabla_i\left(H(|A|^2-\frac{1}{n}H^2)\right),\vec{H}_i^0>$$
$$-\frac{n-1}{2n+1}H|\nabla A|^2+c_{11}H^3(|A|^2-\frac{1}{n}H^2)$$
$$+c_{11}H^3+c_{11}|\nabla A|^2$$

Proof: The results of section 2 give

$$\frac{d}{dt}(|A|^2-\frac{1}{n}H^2)=\Delta(|A|^2-\frac{1}{n}H^2)-<\nabla_l(|A|^2-\frac{1}{n}H^2),\vec{H}_l^0>-2(|\nabla A|^2-\frac{1}{n}|\nabla H|^2)$$
$$+2(|A|^2-\frac{1}{n}H^2)(|A|^2+\bar{R}ic(\nu,\nu))-4(h^{ij}h_j{}^m\bar{R}_{mli}{}^l-h^{ij}h^{lm}\bar{R}_{milj})$$
$$-2h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^l-\bar{\nabla}_l\bar{R}_{0ij}{}^l)+2<h_{ij}^0,V_{ij}>$$

So by lemma 2.11

$$\frac{d}{dt}\left(H(|A|^2-\frac{1}{n}H^2)\right)\leq\Delta\left(H(|A|^2-\frac{1}{n}H^2)\right)-<\nabla_i\left(H(|A|^2-\frac{1}{n}H^2)\right),\vec{H}_i^0>$$
$$-2H(|\nabla A|^2-\frac{1}{n}|\nabla H|^2)+c_9H^3(|A|^2-\frac{1}{n}H^2)+c_9H^3$$
$$+c_{10}H^3(|A|^2-\frac{1}{n}H^2)-2<\nabla_iH,\nabla_i(|A|^2-\frac{1}{n}H^2)>$$

with positive constants c_9 and c_{10}.

As in [H2] we have

(5.0.4) $$-2<\nabla_iH,\nabla_i(|A|^2-\frac{1}{n}H^2)>\leq\ \tilde{c}|\nabla A|^2+\frac{n-1}{2n+1}H|\nabla A|^2$$

with a large constant $\tilde{c}$.

and the proof is finished by lemma 2.12 (b) and corollary 3.3.

Lemma 5.3 : *There exist constants* $c_{12},\ c_{13},\ \tilde{\eta}>0$ *, such that*

(*a*) $\frac{d}{dt}H^3\geq\Delta H^3-<\nabla_iH^3,\vec{H}_i^0>-c_{12}H|\nabla A|^2+\tilde{\eta}H^5$

$$(b) \qquad \tfrac{d}{dt}|A|^2 \leq \Delta|A|^2 - < \nabla_i |A|^2, \overrightarrow{H}^0_i > -2|\nabla A|^2 + c_{13}H^4$$

Proof: (a) follows from 2.8 (d), 2.11, 2.12 and 3.0, whereas (b) is a direct consequence of 2.8 (e) and 2.11.

For a fixed $\eta > 0$ we define the function

$$f_{a_1,a_2} := \frac{|\nabla \widetilde{H}|^2}{\widetilde{H}} + a_1 H(|A|^2 - \frac{1}{n}H^2) + a_2|A|^2 - \eta H^3$$

Lemmata 5.1, 5.2 and 5.3 give the estimate

$$\begin{aligned}\frac{d}{dt} f_{a_1,a_2} &\leq \Delta f_{a_1,a_2} - < \nabla_i f_{a_1,a_2}, \overrightarrow{H}^0_i > + c_8 H|\nabla A|^2 \\ &+ c_8 H^3 - a_1 \frac{n-1}{2n+1} H|\nabla A|^2 + a_1 c_{11} H^3(|A|^2 - \frac{1}{n}H^2) + a_1 c_{11} H^3 \\ &+ a_1 c_{11}|\nabla A|^2 - 2a_2|\nabla A|^2 + a_2 c_{13} H^4 + \eta c_{12} H|\nabla A|^2 - \eta\widetilde{\eta} H^5\end{aligned}$$

That means if

$$a_1 \frac{n-1}{2n+1} > c_8 + \eta c_{12}$$

and

$$2a_2 > a_1 c_{11}$$

, we conclude with theorem 4.0

$$\frac{d}{dt} f_{a_1,a_2} \leq \Delta f_{a_1,a_2} - < \nabla_i f_{a_1,a_2}, \overrightarrow{H}^0_i > + c_{14} H^{5-\sigma} + c_{14} H^4 - \eta\widetilde{\eta} H^5$$

and therefore

$$\frac{d}{dt} f_{a_1,a_2} \leq \Delta f_{a_1,a_2} - < \nabla_i f_{a_1,a_2}, \overrightarrow{H}^0_i > + c_{15}$$

Proof of theorem 5.0

Choose a_1 and a_2 so large that

$$a_1 \frac{n-1}{2n+1} > c_8 + \eta c_{12}$$

$$2a_2 > a_1 c_{11}$$

Then we have

$$\frac{d}{dt} f_{a_1,a_2} \leq \Delta f_{a_1,a_2} - < \nabla_i f_{a_1,a_2}, \overrightarrow{H}_i^0 > + c_{15}$$

and consequently

$$f_{a_1,a_2} \leq c_{16}$$

and

$$|\nabla \widetilde{H}|^2 \leq c_{16} \widetilde{H} + \eta \widetilde{H} H^3$$

By corollary 3.3

$$|\nabla \widetilde{H}|^2 \leq 2\eta d_3 H^4 + c_{17}$$

and (5.0.1) , corollary 3.3 give

$$|\nabla H|^2 \leq 4\eta d_3 H^4 + 2c_{17} + c_5 H^2 \leq 5\eta d_3 H^4 + c_{18}$$

Now the proof of theorem 5.0 follows, since η is arbitrary.

6. The proof of theorem 1.0

Theorem 6.0 :

$$\lim_{t\to T}\max_{M_t}|A|^2=\infty$$

Proof: By [H2], theorem 7.1

$$\lim_{t\to T}\max_{\widetilde{M}_t}|\widetilde{A}|^2=\infty$$

This proves the theorem, if we take into account lemma 3.5, corollary 3.3 and the inequality $H^2\le n|A|^2$.

Exactly as in [H2] we obtain

Lemma 6.1 :

$$\lim_{t\to T}\frac{H_{max}(t)}{H_{min}(t)}=1$$

Lemma 6.2 :

$$\int_0^T H_{max}^2(\tau)d\tau=\infty$$

Lemma 6.3 :

$$\int_0^T h(\tau)d\tau=\infty$$

but here h is defined as

$$h:=\frac{\int_{M_t}H(H+\lambda)d\mu}{\int_{M_t}d\mu}$$

Lemma 6.4 :

$$\lim_{t\to T}\frac{|A|^2}{H^2}=\frac{1}{n}$$

Since the surfaces are shrinking, the last lemmata prove the first part of theorem 1.0. The second part of the theorem then follows with the same calculations as in [S2], since for t close to T the influence of curvature terms corresponding to N are negligible.

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