STARSHAPED HYPERSURFACES AND THE MEAN CURVATURE FLOW

Knut Smoczyk

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Abstract. Under the assumption of two a-priori bounds for the mean curvature, we are able to generalize a recent result due to Huisken and Sinestrari [8], valid for mean convex surfaces, to a much larger class. In particular we will demonstrate that these a-priori bounds are satisfied for a class of surfaces including meanconvex as well as starshaped surfaces and a variety of manifolds that are close to them. This gives a classification of the possible singularities for these surfaces in the case $n = 2$. In addition we prove that under certain initial conditions some of them become mean convex before the first singularity occurs.

0. Introduction

In a recent paper, Huisken and Sinestrari [8] studied the mean curvature flow for mean convex surfaces and proved an estimate on the negative part of the scalar curvature that made it possible to classify all singularities for mean convex surfaces in dimension $n = 2$. The assumption of positive mean curvature had been essential in their work and their method does not carry on to the present context since it relies on some estimates which are only true for mean convex surfaces. In this paper we will prove that the same result is still true, if we assume much weaker assumptions, consisting of two a-priori bounds for the mean curvature (a lower bound for the mean curvature and an upper bound for the squared principal curvatures in terms of a linear function in H^2, i.e. the squared mean curvature). For mean convex surfaces these a-priori estimates are known to be true, but for other surfaces it is generally impossible to verify these a-priori estimates directly since there exist counterexamples, e.g. surfaces for which the minimum of the mean curvature tends to $-\infty$, as is the case for some rotationally symmetric tori developing a singularity on the axis of rotation. Nevertheless, as we will demonstrate in section 2, there are a variety of surfaces including all starshaped surfaces and manifolds that can be obtained by building in small, concave dents into mean convex surfaces, that satisfy our a-priori estimates.

For the convenience of those readers who are familiar with [8], we adapt our notations to the notations used in that paper. Assume that M is an $n-$dimensional closed surface, smoothly immersed into $\mathbb{R}^{n+1}$ by a diffeomorphism F_0 and that

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$F_t : M \to M_t \subset \mathbb{R}^{n+1}$ is a smooth family of diffeomorphisms solving the mean curvature flow (MCF) for $M_t = F_t(M)$, i.e.

$$\frac{\partial}{\partial t} F_t(x) = -H(F_t(x))\nu(F_t(x)).$$

The standard (normal) coordinates for the ambient space, i.e. for $\mathbb{R}^{n+1}$, will be denoted by y^α and coordinates for an immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ with x^i. A summation convention is used; we denote the immersion with F, or F_t when we want to express its dependence on time and we denote the induced metric, second fundamental form, mean curvature and the sum of the squared principal curvatures as usual by $g_{ij}, h_{ij}, H, |A|^2$ resp. In the sequel ∇ always means the induced Levi-Civita connection on the hypersurface and we will write $\langle .,. \rangle$ for the standard euclidean inner product. Regarding the outer unit normal $\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha}$ and the position vector $F = F^\alpha \frac{\partial}{\partial y^\alpha}$ as a set of functions on our hypersurface, we obtain the Gauß-Weingarten-Codazzi equations in the following form:

Proposition 1.

a) $\nabla_i \nabla_j F = -h_{ij}\nu$

b) $\nabla_i \nu = h_i^{\ l} \nabla_l F$

c) $\nabla_l h_{ij} = \nabla_i h_{lj}$

d) $\nabla_i \nabla_j \nu = \nabla^l h_{ij} \nabla_l F - h_i^{\ l} h_{lj} \nu$

e) $\Delta F = -H\nu$

f) $\Delta \nu = \nabla H - |A|^2 \nu$

We recall the evolution equations for various geometric objects on M^n (for details see [5]). Here some of them appear in a slightly different way, since we make use of Prop. 1.e) and f). We have:

Proposition 2.

a) $\frac{\partial}{\partial t} F = \Delta F$

b) $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$

c) $\frac{\partial}{\partial t} \nu = \Delta \nu + |A|^2 \nu$

d) $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{il} h^l_{\ j} + |A|^2 h_{ij}$

e) $\frac{\partial}{\partial t} H = \Delta H + H|A|^2$

f) $\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4$

We will need the evolution equation for another expression.

Proposition 3. *For any fixed* $p \in \mathbb{R}^{n+1}$ *we have*

$$\frac{\partial}{\partial t} \langle F - p, \nu \rangle = \Delta \langle F - p, \nu \rangle + \langle F - p, \nu \rangle |A|^2 - 2H$$

Proof:. In view of Prop. 2 a) and 2 c) we get:

$$\frac{\partial}{\partial t} \langle F - p, \nu \rangle = \langle \Delta(F - p), \nu \rangle + \langle F - p, \Delta\nu + |A|^2 \nu \rangle$$

$$= \Delta \langle F - p, \nu \rangle - 2g^{ij} \langle \nabla_i F, \nabla_j \nu \rangle + \langle F - p, \nu \rangle |A|^2$$

Then Prop. 1 b) and the fact that $g_{ij} = \langle \nabla_i F, \nabla_j F \rangle$ give the result.◇

1. The main result

Throughout this section we will assume that the initial surface is a closed, smooth hypersurface of dimension $n \geq 2$ and that on the maximal time interval $[0, T)$ for which a smooth solution of the MCF exists, the following two a-priori bounds for the mean curvature are valid:

There exist nonnegative constants l, c_0, b, such that

$$H + l \geq 0 \tag{A1}$$

$$|A|^2 \leq c_0 H^2 + b \tag{A2}$$

The aim of this section is to prove our main result:

Theorem 1.1. *Assume that M_t is a family of smooth, closed hypersurfaces evolving by its mean curvature on the maximal time interval $[0, T)$ and that during the evolution process the two a-priori bounds (A1) and (A2) are satisfied. Then for any $0 < \eta \leq 1$ we can also find a constant c, depending on $\eta, M_0, l, c_0, b, n, T$ such that*

$$|A|^2 - H^2 \leq \eta H^2 + c$$

holds for all $t \in [0, T)$.

Lemma 1.1. *If (A1) and (A2) are true for M_t, $t \in [0, T)$, then there exists a positive constant k, such that $\forall t \in [0, T)$*

(i) $H + k > 0$
(ii) $|H| \leq H + k$
(iii) $|A|^2 \leq c_0 (H + k)^2$

Proof:. For any $k_0 > 2l$ we have

$$(H + k_0)^2 - H^2 = k_0(2H + k_0) > 2k_0(H + l) \geq 0$$

and

$$c_0(H + k_0)^2 - (c_0 H^2 + b) = c_0 k_0 (2H + k_0) - b \geq c_0 k_0 (k_0 - 2l) - b \geq 0,$$

$\forall\ k_0 \geq k = k(c_0, l, b)$.$\diamond$

In the forthcoming we set $f := H + k$ with k as in Lemma 1.1.

Lemma 1.2. *Suppose $(1 + \eta) f^2 \leq |A|^2 \leq c_0 f^2$ for some $c_0, \eta > 0$. Then we also have*

a) $-2Z \geq \eta f^2 |A|^2$
b) $Q^2 \geq \frac{\eta^2}{4n(n-1)^2 c_0} f^2 |\nabla f|^2$,

where $Q^2 := |f \nabla_i h_{kl} - \nabla_i f h_{kl}|^2$ *and* $Z := H \operatorname{tr}(A^3) - |A|^4$.

Proof:. Using the Codazzi equation one obtains $Q^2 \geq \frac{1}{4} |\nabla_i f h_{kl} - \nabla_k f h_{il}|^2$ and taking into account Lemma 1.1 we can proceed exactly as in [8]. $\diamond$

We want to control the negative part of the scalar curvature, i.e. the positive part of $|A|^2-H^2$, at those points, where the mean curvature will tend to infinity. Of course, since the mean curvature could vanish somewhere on M_t we cannot devide by H. Therefore, for $\sigma,\eta>0$, we define a similar function as in [8]

$$m_{\sigma,\eta} := \frac{|A|^2-(1+\eta)f^2}{f^{2-\sigma}}$$

Since this expression depends on an extra parameter, namely k, the evolution equation becomes more complicated. An easy calculation shows that

$$\begin{aligned}(1)\qquad \frac{\partial}{\partial t}m &= \Delta m + 2(1-\sigma)f^{-1}\langle\nabla f,\nabla m\rangle - \sigma(1-\sigma)mf^{-2}|\nabla f|^2 - 2f^{\sigma-4}Q^2\\ &\quad+\sigma m|A|^2+(2-\sigma)kmf^{-1}|A|^2+2(1+\eta)kf^{\sigma-1}|A|^2\end{aligned}$$

Now set $m_+ := \max(m,0)$.

Lemma 1.3. *$\forall p\ge 3,\ 0<\sigma<\frac{1}{3}$, we can find positive constants c_1, c_2 depending on k, c_0, p, σ, η such that*

$$\begin{aligned}\frac{\partial}{\partial t}m_+^p &\le \Delta m_+^p - p(p-1)m_+^{p-2}|\nabla m_+|^2 + 2p(1-\sigma)m_+^{p-1}f^{-1}\langle\nabla f,\nabla m_+\rangle\\ &\quad - 2pm_+^{p-1}f^{\sigma-4}Q^2 + 2\sigma pm_+^p|A|^2 + c_1m_+^p + c_2\end{aligned}$$

Proof:. In view of (1) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}m_+^p &\le \Delta m_+^p - p(p-1)m_+^{p-2}|\nabla m_+|^2 + 2p(1-\sigma)m_+^{p-1}f^{-1}\langle\nabla f,\nabla m_+\rangle\\ &\quad - 2pm_+^{p-1}f^{\sigma-4}Q^2 + \sigma pm_+^p|A|^2 + 2p(1+\eta)kf^{\sigma-1}m_+^{p-1}|A|^2\\ &\quad + pk(2-\sigma)m_+^pf^{-1}|A|^2\end{aligned}$$

Using Schwartz´ inequality we get

$$pk(2-\sigma)m_+^pf^{-1}|A|^2 \le pk(2-\sigma)m_+^p\left(\frac{|A|^2}{2\epsilon}+\frac{\epsilon|A|^2}{2f^2}\right),\ \forall\epsilon>0$$

Now choose $\epsilon := \frac{k(2-\sigma)}{\sigma}$. Since $|A|^2\le c_0f^2$, we obtain

$$(2)\qquad pk(2-\sigma)m_+^pf^{-1}|A|^2 \le \frac{\sigma}{2}pm_+^p|A|^2 + pc_0\frac{k^2(2-\sigma)^2}{2\sigma}m_+^p$$

We also have in view of Lemma 1.1 and $\sigma<1$, that

$$2p(1+\eta)kf^{\sigma-1}m_+^{p-1}|A|^2 \le 2p(1+\eta)kc_0^{\frac{1-\sigma}{2}}|A|^{1+\sigma}m_+^{p-1}$$

Using Young's inequality $xy\le\delta x^r+\delta^{-\frac{q}{r}}y^q$, with $x = m_+^{\frac{1+\sigma}{1-\sigma}}|A|^{1+\sigma}$, $y=1$ and $\delta = \frac{\sigma}{4(1+\eta)k}c_0^{\frac{\sigma-1}{2}}, r=\frac{2}{1+\sigma},\ q=\frac{2}{1-\sigma}$, we get

$$2p(1+\eta)kc_0^{\frac{1-\sigma}{2}}|A|^{1+\sigma}m_+^{p-1} \le \frac{\sigma}{2}pm_+^p|A|^2+\frac{\sigma}{2}p\delta^{-\frac{2}{1-\sigma}}m_+^{p-\frac{2}{1-\sigma}}$$

For any $q_1 > q_2 > 0$ we can find a constant $c > 0$ depending only on q_1 and q_2, such that $\forall x \geq 0: \ x^{q_2} \leq x^{q_1} + c$. With $q_1 = p, \ q_2 = p - \frac{2}{1-\sigma}$, we conclude

$$(3) \qquad 2p(1+\eta)kf^{\sigma-1}m_+^{p-1}|A|^2 \leq \frac{\sigma}{2}pm_+^p|A|^2 + \frac{\sigma}{2}p\delta^{-\frac{2}{1-\sigma}}(m_+^p + c(\sigma,p))$$

Inequalities (2) and (3) give

$$(4) \quad 2p(1+\eta)kf^{\sigma-1}m_+^{p-1}|A|^2 + pk(2-\sigma)m_+^p f^{-1}|A|^2 \leq \sigma pm_+^p|A|^2 + c_1 m_+^p + c_2 \ ,$$

where c_1, c_2 depend on p, σ, η, c_0, k. This completes the proof. ◇

Lemma 1.4. *Let* $c_3 := 4n(n-1)^2c_0\eta^{-2}$. *Then* $\forall p \geq \max(3, 1+4c_0c_3), \ 0 < \sigma < \frac{1}{3}$, *we can find positive constants* c_1, c_4 *depending only on* $k, c_0, p, \sigma, \eta, M_0$ *such that*

$$\frac{\partial}{\partial t}\int m_+^p d\mu \leq -\frac{p(p-1)}{2}\int m_+^{p-2}|\nabla m_+|^2 d\mu - p\int m_+^{p-1}f^{\sigma-4}Q^2 d\mu$$
$$- \frac{p}{2c_3}\int m_+^{p-1}f^{\sigma-2}|\nabla f|^2 d\mu + 2\sigma p\int m_+^p|A|^2 d\mu + c_1\int m_+^p d\mu + c_4$$

Proof:. From Lemma 1.2 we deduce

$$(5) \qquad m_+^{p-1}f^{\sigma-4}Q^2 \geq \frac{\eta^2}{4n(n-1)^2c_0}f^{\sigma-2}m_+^{p-1}|\nabla f|^2$$

and with $c_3 := 4n(n-1)^2c_0\eta^{-2}$ and Schwartz´ inequality we get

$$(6) \quad 2(1-\sigma)pm_+^{p-1}f^{-1}\langle\nabla f, \nabla m_+\rangle \leq pm_+^{p-1}f^{\sigma-4}Q^2 - \frac{p}{2c_3}m_+^{p-1}f^{\sigma-2}|\nabla f|^2$$
$$+ \frac{p(p-1)}{2}m_+^{p-2}|\nabla m_+|^2,$$

$\forall \ p \geq \max(2, 1+4c_0c_3), \ \sigma < 1$.
Now we set $c_4 := c_2|M_0|$. Then Lemma 1.4 follows from (6), Lemma 1.3 and the fact that $|M_t|$ is decreasing. ◇

An easy calculation shows that

$$(7) \quad \Delta m = 2f^{\sigma-2}\langle h_{ij}, \nabla_i\nabla_j f\rangle + 2f^{\sigma-2}Z - ((2-\sigma)mf^{-1} + 2(1+\eta)f^{\sigma-1})\Delta f$$
$$+ 2f^{\sigma-4}Q^2 + \sigma(1-\sigma)mf^{-2}|\nabla f|^2 - 2(1-\sigma)f^{-1}\langle\nabla m, \nabla f\rangle$$

Multiplying this with $m_+^p f^{-\sigma}$ gives after partial integration and use of the Codazzi equations

$$(8) \quad -2\int m_+^p f^{-2}Z d\mu = p\int m_+^{p-1}f^{-\sigma}|\nabla m_+|^2 d\mu + 4m_+^p f^{-3}\langle\nabla_i f\nabla_j f, h_{ij}\rangle d\mu$$
$$+ 2\int f^{-4}m_+^p Q^2 d\mu - 2p\int m_+^{p-1}f^{-2}\langle\nabla_i m_+\nabla_j f, h_{ij}\rangle d\mu$$
$$+ \int\langle\nabla f, \nabla m_+\rangle(p(2-\sigma)m_+^p f^{-\sigma-1} + 2(1+\eta)pm_+^{p-1}f^{-1})d\mu$$
$$- 2\int|\nabla f|^2((2+\eta)m_+^p f^{-2} + m_+^{p+1}f^{-2-\sigma})d\mu$$

Without loss of generality we can assume that $c_0 \geq 1$. Then Lemma 1.1 and (8) yield in view of Lemma 1.2 a) that for $p \geq 1,\ \sigma < 1,\ \eta \leq 1$

$$(9)\ \eta \int m_+^p |A|^2 d\mu \leq pc_0 \int m_+^{p-2} |\nabla m_+|^2 + 4p(1+c_0) \int m_+^{p-1} f^{-1} |\nabla m_+||\nabla f| d\mu \\ + 4c_0^2 \int m_+^{p-1} f^{2-\sigma} |\nabla f|^2 d\mu + 2c_0 \int f^{\sigma-4} m_+^{p-1} Q^2 d\mu$$

Then we use Schwartz´ inequality for the second term on the Righthandside and obtain for any $\beta > 0$ the inequality

$$(10)\quad \frac{\eta}{4c_0^2} \int m_+^p |A|^2 d\mu \leq p(1+\frac{1}{\beta}) \int m_+^{p-2} |\nabla m_+|^2 \\ + (p\beta + 1) \int m_+^{p-1} f^{2-\sigma} |\nabla f|^2 d\mu + \int f^{\sigma-4} m_+^{p-1} Q^2 d\mu$$

Now assume that $c_5 := \frac{4c_0^2}{\eta}$ and that $(\sigma,\ p)$ is any pair such that

$$0 < \sigma \leq \min(1, \frac{1}{2c_5}\sqrt{\frac{1}{24c_3 p}})\ ,\ p \geq \max(3, \frac{3}{2c_3}, \frac{2c_3}{3})$$

and choose $\beta := 12c_5\sigma$. We conclude

$$2\sigma p^2 c_5 (1+\frac{1}{\beta}) = p^2(\frac{1}{6} + 2c_5\sigma) \leq \frac{1}{3}p^2 \leq \frac{p(p-1)}{2}$$
$$2\sigma c_5 p(p\beta + 1) = 6(2\sigma c_5)^2 p^2 + 2\sigma c_5 p \leq \frac{p}{4c_3} + p\sqrt{\frac{1}{24c_3 p}} \leq \frac{p}{2c_3}$$
$$2\sigma p c_5 \leq p$$

As an immediate consequence of these estimates and (10) we obtain

Lemma 1.5. *There exist constants $c_6,\ c_7$, depending only on η, c_0, n such that for all pairs (σ, p) with $p \geq c_6,\ \sigma \leq \frac{c_7}{\sqrt{p}}$*

$$2\sigma p \int m_+^p |A|^2 d\mu \leq \frac{p(p-1)}{2} \int m_+^{p-2} |\nabla m_+|^2 d\mu \\ + \frac{p}{2c_3} \int m_+^{p-1} f^{2-\sigma} |\nabla f|^2 d\mu + p \int f^{\sigma-4} m_+^{p-1} Q^2 d\mu$$

We are now able to show that high L^p-norms of $m_{\sigma,\eta}$ are bounded for suitable pairs (p, σ).

Lemma 1.6. *There are constants c_6, c_7, depending only on η, c_0, n such that for any pair (p, σ) with $p \geq c_6,\ \sigma \leq \frac{c_7}{\sqrt{p}}$ we can find a constant c depending on $M_0, p, \sigma, k, \eta, c_0, T$ such that*

$$\int m_+^p d\mu \leq c\ ,\ \forall t \in [0, T)$$

Proof:. Lemma 1.4 and Lemma 1.5 imply that

$$\frac{\partial}{\partial t}\int m_+^p d\mu \le c_1 \int m_+^p d\mu + c_4$$

for constants c_1, c_4 depending on $k, c_0, p, \sigma, M_0, \eta$. Therefore

$$\int m_+^p d\mu \le (\int m_+^p d\mu_{|t=0} + \frac{c_4}{c_1})e^{c_1 t} - \frac{c_4}{c_1} \le c(T, k, \eta, p, \sigma, M_0) < \infty$$

since $T < \infty$ for closed hypersurfaces. ⋄

Remark. This bound is worse than in [8] since the RHS depends on p and σ and actually tends to $+\infty$ for $\sigma \to 0$.

Corollary 1.7. *For any* $p \ge \max(c_6, \frac{4l^2}{c_7^2})$, $\sigma \le \frac{c_7}{2\sqrt{p}}$ *we have*

$$\int f^l m_{\sigma,\eta}^p d\mu \le c$$

with a constant c *depending on* $\sigma, p, T, c_0, k, \eta, M_0, l$.

Proceeding as in [8] we can show that $m_{\sigma,\eta}$ remains bounded for any $\eta \le 1$, if σ is significantly small. This completes the proof of Theorem 1.1.

Since (A1) implies a lower bound for the mean curvature, the above result and slight modifications of the rescaling techniques used in [8] then also prove the same singularity behavior of surfaces satisfying (A1) and (A2), i.e. we have

Theorem 1.2. *Let* $M_t^{n\ge 2}$ *be a smooth solution of the MCF and assume that the initial manifold* M_0 *is closed and that* M_t *satisfies the a-priori estimates (A1) and (A2) and develops a type II singularity for* $t \to T$. *Then a subsequence of a rescaled flow converges smoothly to an eternal solution* $\widetilde{M}_\tau$ *of the MCF. Either* $\widetilde{M}_\tau$ *has positive scalar curvature everywhere or (up to a rigid motion)* $\widetilde{M}_\tau = \mathbb{R}^{n-1} \times \bar{\Gamma}_\tau$, *where* $\bar{\Gamma}_\tau$ *is the "grim reaper" curve given by* $x = -\ln\cos y + \tau$. *If in addition* $n = 2$, *then the only possible limiting flows under the rescaling procedure in* [8] *are either* $S^2, \mathbb{R} \times S^1, \mathbb{R} \times \Gamma$ *in the case of a type I singularity or in the case of a type II singularity are given by a strictly convex eternal solution or by* $\mathbb{R} \times \bar{\Gamma}$, *where* Γ *are the selfsimilar curves studied by Abresch-Langer* [1] *and* $\bar{\Gamma}$ *is the "grim reaper" curve introduced by Hamilton* [4].

2. Surfaces that satisfy the a-priori estimates

In this section we are going to prove that there exist surfaces, including not only all mean convex but also any starshaped surface and a variety of manifolds with small dents, that satisfy the a-priori estimates (A1) and (A2).

Theorem 2.1. *Let* M_0 *be a smoothly immersed, closed hypersurface such that there exist a point* $p \in \mathbb{R}^{n+1}$ *and nonnegative constants* $a_1, a_2 \in \mathbb{R}$, $a_1 + a_2 > 0$ *with*

$$a_1 H + a_2 \langle F - p, \nu\rangle \ge 0$$

Then we can find constants c_0, b, l *such that (A1) and (A2) hold during the evolution of* M_t *for* $t \in [0, T)$.

Remark. For mean convex surfaces we can choose $a_1 = 1$ and $a_2 = 0$, whereas for starshaped surfaces or more generally starshapedly immersed surfaces, i.e. surfaces for which $\langle F-p,\nu\rangle > 0$ for a point $p \in \mathbb{R}^{n+1}$ (e.g. the limaçon of Pascal, given in polar coordinates by $r = 1+2\cos\theta$), we can choose $a_1 = 0$ and $a_2 = 1$. If M_0 is a mean convex torus in $\mathbb{R}^3$, rotationally symmetric w.r.t to the $z-$axis and symmetric w.r.t. the $(x,y)-$plane, then we can choose p to be the origin in $\mathbb{R}^3$ and let a_2 be a sufficiently small positive constant such that the above inequality is satisfied with $a_1 = 1$. If we now build a small concave dent into the region of the torus, where the radius is maximal, this inequality is still preserved, since $\langle F,\nu\rangle$ is positive on that domain. This example is neither starshapedly immersed nor is it mean convex, but it still satisfies the above estimate.

To prove Theorem 2.1 we first need some Propositions

Proposition 4. *Let* $h := (2a_2t+a_1)H + a_2\langle F-p,\nu\rangle$. *Then* h *satisfies*

$$\frac{\partial}{\partial t}h = \Delta h + |A|^2 h$$

Proof:. From the evolution equations for H (see Prop. 2 e)) and from Proposition 3 we obtain

$$\frac{\partial}{\partial t}h = (2a_2t+a_1)(\Delta H + H|A|^2) + 2a_2H + a_2(\Delta\langle F-p,\nu\rangle + \langle F-p,\nu\rangle|A|^2 - 2H)$$

$$= \Delta h + h|A|^2\ .\diamond$$

Proposition 5. *We have*

$$\frac{\partial}{\partial t}\frac{|A|^2}{h^2} = \Delta\frac{|A|^2}{h^2} - 2\frac{|h\nabla_i h_{jk} - \nabla_i h h_{jk}|^2}{h^4} + \frac{2}{h}\langle\nabla\frac{|A|^2}{h^2},\nabla h\rangle$$

Proof:. From Propostions 2 f) and 4 we obtain

$$\frac{\partial}{\partial t}\frac{|A|^2}{h^2} = \frac{1}{h^2}\Delta|A|^2 - \frac{2}{h^2}|\nabla A|^2 - \frac{2|A|^2}{h^3}\Delta h$$

On the other hand we deduce

$$-\frac{2}{h^2}|\nabla A|^2 = -2\frac{|h\nabla_i h_{jk} - \nabla_i h h_{jk}|^2}{h^4} + 2\frac{|A|^2}{h^4}|\nabla h|^2 - \frac{2}{h^3}\langle\nabla h,\nabla|A|^2\rangle$$

and

$$\Delta\frac{|A|^2}{h^2} = \frac{1}{h^2}\Delta|A|^2 - \frac{2|A|^2}{h^3}\Delta h + 6\frac{|A|^2}{h^4}|\nabla h|^2 - \frac{4}{h^3}\langle\nabla|A|^2,\nabla h\rangle$$

Inserting these two equations gives the result.

Proof of Theorem 2.1:. From Proposition 4 and the strong parabolic maximum principle we deduce that $h > 0$ for all $t \in [\epsilon, T)$, where $0 < \epsilon < T$. Therefore $\frac{|A|^2}{h^2}$ is well-defined for $t \in [\epsilon, T)$ and again by the parabolic maximum principle and Proposition 5 we deduce that $\frac{|A|^2}{h^2}$ is bounded above by $\max\limits_{M_\epsilon}\frac{|A|^2}{h^2}$ and we conclude that

$$|A|^2 \le \max_{M\times[0,\epsilon]}|A|^2 + h^2\max_{M_\epsilon}\frac{|A|^2}{h^2}$$

Now we take into account that by a comparison principle we have $T < \infty$ and that $|F|$ is also bounded above. By definition of h it then follows that

$$|A|^2 \leq c_0 H^2 + b$$

with two nonnegative constants c_0, b depending only on M_0, T. On the other hand we deduce with the same argument and the positivity of $2a_2 t + a_1$ for $t > 0$, that $h \geq 0$ implies the existence a constant $l \geq 0$ such that $H + l \geq 0$ for all $t \in [0, T)$. This completes the proof of Theorem 2.1. $\diamond$

In the remainder we want to show that under certain initial conditions a surface can become mean convex before the first singularity occurs. To this end let us define the set

$$C := \{p \in \mathbb{R}^{n+1} | \ \exists \ a_1, a_2 \geq 0, a_1 + a_2 > 0\,, \text{ with } a_1 H + a_2 \langle F - p, \nu \rangle > 0 \text{ on } M_0\}$$

From the definition of C and the smoothness of F it immediately follows that C is a, possibly empty, open convex domain in $\mathbb{R}^{n+1}$.

Lemma. *For $t \in [0, T)$ we have*

$$H > 0, \ \forall x \in M_t \cap C$$

and

$$H \geq 0, \ \forall x \in M_t \cap \bar{C}$$

Proof:. Let $p \in C \cap M_t$ and h as in Proposition 4. Then we have $h > 0$ on M_t for all $t \in [0, T)$ but in p we have that $h = (2a_2 t + a_1)H$ and since $h > 0, a_1, a_2 \geq 0$ and $a_1 + a_2 > 0$ we deduce $H > 0$. The second statement in the above Lemma is obvious. $\diamond$

Theorem 2.2. *Assume that for $R > 0, a > 1$ there exists a point $p \in C$ such that $B_R(p) \subset C$ and $M_0 \subset B_{aR}(p)$, where $B_R(p)$ denotes the open ball of radius R around p. Further assume that $\max_{M_0} |A|^2 < \dfrac{n}{(a^2-1)R^2}$. Then $T > \dfrac{(a^2-1)R^2}{2n}$ and for all t with $T > t > \dfrac{(a^2-1)R^2}{2n}$ we have $H > 0$ on M_t.*

Proof:. From Proposition 2 f) and the maximum principle we deduce

$$|A|^2 \leq \frac{1}{(\max_{M_0} |A|^2)^{-1} - 2t}$$

and since $\max |A|^2 \to \infty$ for $t \to T$ we conclude that

$$T \geq \frac{1}{2 \max_{M_0} |A|^2} > \frac{(a^2-1)R^2}{2n}$$

It is well-known that two hypersurfaces that do not intersect each other must stay disjoint during their evolution. This implies that M_t lies in the interior of $B_r(p)$, where $r = r(t)$ is the radius for the family of spheres evolving by mean curvature such that $r(0) = aR$. $r(t)$ is explicitly given by $r(t) = \sqrt{(aR)^2 - 2nt}$ and this gives for $t > \frac{(a^2-1)R^2}{2n}$ that $r(t) < R$ and $M_t \subset B_{r(t)}(p) \subset B_R(p) \subset C$. The Theorem then follows from the above Lemma. $\diamond$

Remark:. It is unclear, if our a-priori estimate (A2) already follows from (A1). In that case the above results would provide a classification of all singularities for $2-$dimensional closed surfaces in $\mathbb{R}^3$ with mean curvature bounded below.

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Mathematics Department, ETH Zürich, Switzerland
E-mail address: `ksmoczyk@math.ethz.ch`