Harnack inequality for the Lagrangian mean curvature flow

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Abstract

If L^n is a Lagrangian manifold immersed into a Kähler-Einstein manifold, nothing is known about its behavior under the mean curvature flow. As a first result we derive a Harnack inequality for the mean curvature potential of compact Lagrangian immersions L^n immersed into $\mathbb{R}^{2n}$.

1 Introduction

The mean curvature flow (MCF) for hypersurfaces in arbitrary Riemannian manifolds is well understood whereas almost nothing is known when the codimension is greater than 1. The main problem in the study of immersions with higher codimension is the fact that in general there is no canonical choice of a field of frames in the normal bundle of the immersion and therefore the investigation of the second fundamental form is a delicate problem. However, in some situations it is possible to identify the tangent space of a submanifold in a unique manner with its normal space. An example are the Lagrangian submanifolds. The geometry of the ambient space then must guarantee that the MCF preserves the Lagrangian structure. It turns out that this is true if the ambient space is Kähler-Einstein [10]. Harnack inequalities have always been of great interest in the study of partial differential equation (e.g. see [12]). Many interesting results are known for geometric flow problems [1], [4], [5]. In this paper we will derive a Harnack inequality for the Lagrangian angle in the situation of an immersed Lagrangian manifold in $\mathbb{R}^{2n}$. To explain our result we first recall some terminology and definitions.

Let L^n be a smooth manifold, immersed into a Kähler-Einstein manifold M^{2n} with complex structure J, curvature K and metric $\overline{g}$. Let $\overline{\omega} = \overline{g}(J\cdot, \cdot)$ be the Kählerform on M and let g, ω denote the pullbacks to L. If $\omega = 0$, then L is

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called Lagrangian or totally real. Given a $1-$form on a Lagrangian immersion, one can first use the metric g to identify this $1-$form with a vector field and then apply the complex structure which by the Lagrangian condition maps this tangent vector field to a normal vector field. Assuming that we deform the Lagrangian manifold in this direction one obtains the necessary condition that this $1-$form has to be closed in order to maintain the Lagrangian structure. It is possible to define a $1-$form in terms of the second fundamental form and we denote this $1-$form by "mean curvature form" (see definition below) since the resulting deformation vector field is given by the mean curvature vector which can be defined for arbitrary smooth immersions. It is then an easy consequence of the Codazzi equations and a well-known fact that this mean curvature form is closed and therefore an infinitesimal symplectic motion, if the ambient space is Kähler-Einstein. One can prove that this is not only an infinitesimal but also an actual deformation (see [10]).

2 Notations

We define the following tensor on L:

$$h(u,v,w) := -\overline{g}(J(u), \overline{\nabla}_v w) = \overline{g}(\overline{\nabla}_v J(u), w)$$

where $\overline{\nabla}$ denotes the covariant derivative on M. For a fixed vector u this is the second fundamental form with respect to the normal vector $J(u)$. Assume that $F : L^n \to M^{2n}$ is an immersion, that $x^i,\ y^\alpha,\ i = 1 \dots n,\ \alpha = 1 \dots 2n$ are coordinates for $L^n,\ M^{2n}$ respectively and set $e_i := \frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}$, where double indices are always summed from 1 to n or to $2n$ respectively. Further we will always write $\langle u, v\rangle$ instead of $\overline{g}(u,v)$. Then

$$g_{ij} = \overline{g}_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}$$

$$\overline{\omega}_{\alpha\beta} = \overline{g}_{\beta\gamma} J^\gamma{}_\alpha$$

$$\overline{\omega}_{\alpha\beta} = -\overline{\omega}_{\beta\alpha}$$

$$\overline{\nabla} J = 0,\ J^2 = -\mathrm{Id},\ \langle J(v), J(w)\rangle = \langle v, w\rangle$$

Since $h(u,\cdot,\cdot)$ is the second fundamental form with respect to $J(u)$, we clearly have

$$h(e_k, e_i, e_j) = h_{kij} = h_{kji}$$

On the other hand the properties of J imply that this tensor is also symmetric in the other two indices. In the forthcoming we will set $e_i^\alpha := \frac{\partial F^\alpha}{\partial x^i}$ and $\nu_s := J(e_s)$. We therefore have the Gauss-Weingarten-Codazzi equations:

Proposition 2.1

$$h_{ijk} = h_{jik} = h_{jki} \tag{7}$$

$$\frac{\partial e_k^\gamma}{\partial x^j} - \Gamma_{kj}^n e_n^\gamma + \overline{\Gamma}_{\alpha\beta}^\gamma e_k^\alpha e_j^\beta = -h^n{}_{kj} \nu_n^\gamma \tag{8}$$

$$\frac{\partial \nu_s^\gamma}{\partial x^i} - \Gamma^l_{is}\nu_l^\gamma + \overline{\Gamma}^\gamma_{\alpha\beta} e_i^\alpha \nu_s^\beta = h^n{}_{is} e_n^\gamma \tag{9}$$

$$\nabla_l h_{kji} - \nabla_k h_{lji} = \overline{R}_{lkj\underline{i}} \tag{10}$$

$$R_{ijkl} = \overline{R}_{ijkl} + g^{mn}(h_{mik}h_{njl} - h_{mil}h_{njk}) \tag{11}$$

$$h_{jkl} = \overline{\omega}_{\alpha\beta} e^\alpha_{jk} e^\beta_l + \overline{\omega}_{\delta\gamma}\overline{\Gamma}^\delta_{\alpha\beta} e^\alpha_j e^\beta_k e^\gamma_l \tag{12}$$

where an underlined index means that one has to take the image of this vector under J, e.g. $\overline{R}_{\underline{i}jkl} = \langle \overline{R}(e_k, e_l)e_j, J(e_i)\rangle$

Definition 2.2

$$H := H_i dx^i := g^{kl} h_{ikl} dx^i$$

is called the mean curvature form on L.

If in addition L is immersed into a Kähler-Einstein manifold then the trace of the Codazzi equation (10) gives the identity

$$dH = 0 \tag{14}$$

The Lagrangian MCF equation now takes the form

$$\frac{\partial}{\partial t} F^\alpha = -g^{mn} H_m J^\alpha{}_\beta \frac{\partial F^\beta}{\partial x^n} = -H^n \nu_n^\alpha \tag{$*$}$$

In [10] we proved that the evolution equations for the metric, the second fundamental form and the mean curvature form are given by

Lemma 2.3

$$\frac{\partial}{\partial t} g_{ij} = -2H^l h_{lij} \tag{15}$$

$$\frac{\partial}{\partial t} h_{jkl} = \nabla_k \nabla_j H_l - H^n({h_{nj}}^m h_{mkl} + {h_{nl}}^m h_{mkj}) + H^n \overline{R}_{\underline{n}k\underline{l}j} \tag{16}$$

$$\frac{\partial}{\partial t} H = dd^\dagger H + KH \tag{17}$$

Here $d^\dagger$ denotes the negative adjoint to d, i.e. $d^\dagger H = \nabla^i H_i = g^{ij}\nabla_i H_j$. In particular for the form $\widetilde{H} := e^{-tK} H$ we obtain the evolution equation

$$\frac{\partial}{\partial t}\widetilde{H} = dd^\dagger \widetilde{H} \tag{18}$$

Lemma 2.4 (Representation formula) *Assume that $L_t = F_t(L)$ is compact, orientable and evolves under the MCF and that x_0 is an arbitrary but fixed point on L and denote the initial mean curvature form by H_0.*

(a) There exists a unique smooth family of functions ϕ, smoothly depending on time such that

$$\widetilde{H} = H_0 + d\big(\int_0^t \Delta\phi d\tau\big)$$

$$\Delta\big(\phi - \int_0^t \Delta\phi d\tau\big) = d^\dagger H_0$$

$$\phi(x_0) = 0$$

in particular the cohomology class of $\widetilde{H}$ does not change.

(b) If H_0 is exact, then there exists a unique smooth family of functions ϕ such that

$$\widetilde{H} = d\phi$$

$$\frac{\partial}{\partial t}\phi = \Delta\phi$$

$$\min_L \phi_0 = 0$$

(c) If $S \subset L$ is simply connected then there exist a unique smooth family of functions ϕ on S such that

$$\widetilde{H} = d\phi$$

$$\frac{\partial}{\partial t}\phi = \Delta\phi$$

$$\min_S \phi_0 = 0$$

Proof: Define the form $\widehat{H}_t(x) := H_0(x) + d\big(\int_0^t d^\dagger \widetilde{H} d\tau\big)$ where we integrate pointwise. This form surely exists, since $d^\dagger \widetilde{H}$ is smooth. For the time derivative we obtain

$$\frac{\partial}{\partial t}\widehat{H} = dd^\dagger \widetilde{H} = \frac{\partial}{\partial t}\widetilde{H}$$

and since $\widehat{H}_0 = \widetilde{H}_0 = H_0$ we conclude $\widehat{H} = \widetilde{H}$. Now we use the decomposition theorem and can express $\widetilde{H}$ as a unique sum $\widetilde{H} = \psi + d\phi$, where $d^\dagger\psi = 0$, $\phi(x_0) = 0$ and $\psi,\ \phi$ are smooth. Then $d^\dagger \widetilde{H} = d^\dagger d\phi = \Delta\phi$. This proves (a). (b) is a direct consequence of (a) since then the harmonic part $\psi \equiv 0$ and therefore

$$\widetilde{H} - H_0 = d(\phi_t - \phi_0) = d\big(\int_0^t \Delta\phi d\tau\big)$$

This implies

$$d\big(\int_0^t \frac{\partial}{\partial t}\phi - \Delta\phi d\tau\big) = 0$$

which means that there exists a smooth function $f(t)$ such that $\frac{\partial}{\partial t}\phi - \Delta\phi = f(t)$. Now define $\widetilde{\phi} := \phi - \int_0^t f d\tau - min_L \phi_0$. This function has all the desired properties. If $\phi,\ \widetilde{\phi}$ are two functions with the same properties then $d(\phi - \widetilde{\phi}) = 0$

and consequently there exists a function $f(t)$ such that $\phi = \widetilde{\phi} + f(t)$. Since $\frac{\partial}{\partial t}(\phi - \widetilde{\phi}) = \Delta(\phi - \widetilde{\phi}) = 0$ we conclude that this function is a constant c which has to be zero since $\min_L \phi_0 = \min_L \widetilde{\phi}_0 = 0$. This proves uniqueness. (c) follows in the same way as (b) since $\widetilde{H}$ is closed and therefore exact on any simply connected $S \subset L$. □

In view of part (b) we make the following definition

Definition 2.5 *A solution for the Lagrangian MCF for which the initial mean curvature form is exact will be called an exact solution.*

So for an exact solution to the MCF we obtain a smooth family of mean curvature potentials $\widetilde{\phi}$ with $H = d\widetilde{\phi}$ and $\frac{\partial}{\partial t}\widetilde{\phi} = \Delta\widetilde{\phi} + K\widetilde{\phi}$ which is unique up to adding a constant multiple of e^{Kt}. In particular the mean curvature vector for a closed exact solution vanishes always at least in two points.

3 The result

During this section we will assume that the ambient space is $\mathbb{R}^{2n}$ with its canonical complex structure and that L^n is a compact Lagrangian immersion with exact mean curvature form.

Remark: *In this context it is natural to ask wether there exist Lagrangian immersions with exact mean curvature form. If the target manifold M is arbitrary then any Lagrangian immersion of the sphere S^n with $n > 1$ or any other simply connected manifold has exact mean curvature form since the first Betti number vanishes in that case. On the other hand a figure eight curve in $\mathbb{C}$ has total curvature zero, thus H must be exact. In addition any torus obtained as a cross product of n different figure eight curves in $\mathbb{C}$ viewed as a Lagrangian immersion in $\mathbb{C}^n$ also has exact mean curvature form.*

Assume that $f : L \times [0, t_0)$ is a smooth positive solution of the "heat" equation

$$\frac{\partial f}{\partial t} = \Delta_t f \qquad (**)$$

where Δ_t is the Laplace Beltrami Operator w.r.t. the metric $g(\cdot, t)$. Solutions of $(**)$ appear in abundance, the mean curvature potential is one example (we will give more examples in the remark below) and therefore it is important to investigate them. Let us define the tensors

$$\begin{aligned} a_{ij} &:= H^n h_{nij} \\ b_{ij} &:= h_i{}^{mn} h_{mnj} \end{aligned}$$

The first result is quite general

Theorem 3.1 *Let f be a positive solution of (**) with $f \leq A$ and let $B > 0$ be an upper bound for the tensor b_{ij}, i.e. $b_{ij}V^iV^j \leq B|V|^2$, $\forall V \in TL$. Then*

$$(1 - e^{-2Bt})|\nabla f|^2 \leq 2Bf^2 \ln(\frac{A}{f}) \tag{31}$$

Proof: We compute

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla f|^2 &= 2a^{ij}\nabla_i f \nabla_j f + 2\nabla^i f \nabla_i \Delta f \\
&= 2a^{ij}\nabla_i f \nabla_j f + \Delta|\nabla f|^2 - 2|\nabla_i \nabla_j f|^2 - 2R_{ij}\nabla^i f \nabla^j f \\
&= \Delta|\nabla f|^2 - 2|\nabla_i \nabla_j f|^2 + 2b_{ij}\nabla^i f \nabla^j f
\end{aligned}$$

and then

$$\frac{\partial}{\partial t}\frac{|\nabla f|^2}{f} = \Delta\frac{|\nabla f|^2}{f} - \frac{2}{f}|\nabla_i\nabla_j f - \frac{\nabla_i f \nabla_j f}{f}|^2 + \frac{2}{f}b_{ij}\nabla^i f \nabla^j f \tag{32}$$

We also have

$$\frac{\partial}{\partial t}(f \ln(\frac{A}{f})) = \Delta(f\ln(\frac{A}{f})) + \frac{|\nabla f|^2}{f}$$

Let $p := \frac{1}{2B}(1 - e^{-2Bt})$. Then $\frac{\partial}{\partial t}p = 1 - 2Bp$ and we obtain for

$$h := p\frac{|\nabla f|^2}{f} - f\ln(\frac{A}{f})$$

$$\begin{aligned}
\frac{\partial}{\partial t}h &= \Delta h - \frac{2p}{f}|\nabla_i\nabla_j f - \frac{\nabla_i f\nabla_j f}{f}|^2 + (\frac{\partial}{\partial t}p - 1)\frac{|\nabla f|^2}{f} + \frac{2p}{f}b_{ij}\nabla^i f\nabla^j f \\
&\leq \Delta h + (\frac{\partial}{\partial t}p - 1 + 2Bp)\frac{|\nabla f|^2}{f} = \Delta h
\end{aligned}$$

and the result follows from the maximum principle since at $t = 0$ we have $h \leq 0$.
□

Remark: *The condition that $b_{ij} - Bg_{ij}$ is negative semidefinite is true on any compact time interval $[0, T_0]$ on which a smooth solution of (*) exists as long as we choose a large constant B. However a uniform bound for b_{ij} is mostly not given and not expected. It is natural that the eigenvalues of b_{ij} should tend to infinity whenever a singularity occurs during the flow. In special situations, like for Lagrangian graphs in $\mathbb{C}^n$, we expect the tensor b_{ij} to be uniformly bounded in t and that these graphs converge to a minimal Lagrangian immersion.*

Corollary 3.2 *There exists a constant C depending only on $L_0 = F_0(L^n)$ such that for any positive solution f of (**) and $0 < t \leq 1$ we have*

$$f(x,t) \leq \frac{C}{t^{n/2}}\int_L f(x,t)d\mu_t(x) \tag{35}$$

Proof: This follows in the same way as in [3], if we take into account that

$$\frac{\partial}{\partial t}\int_L f d\mu_t = \int_L \Delta f - f|H|^2 d\mu_t = -\int_L f|H|^2 d\mu_t \leq 0$$

□

Lemma 3.3 *Let f be a solution of (**). Then we have*

$$(\frac{\partial}{\partial t} - \Delta)\frac{\partial}{\partial t}f = 2h^{kij}\nabla_i H_j \nabla_k f + 2a^{ij}\nabla_i\nabla_j f \tag{37}$$

Proof: For the time derivative of the connection Γ_{ij}^k one has

$$\begin{aligned} -g^{ij}\frac{\partial}{\partial t}\Gamma_{ij}^k &= -\frac{1}{2}g^{ij}g^{kl}(\nabla_i(\frac{\partial}{\partial t}g_{lj}) + \nabla_j(\frac{\partial}{\partial t}g_{li}) - \nabla_l(\frac{\partial}{\partial t}g_{ij})) \\ &= g^{ij}g^{kl}(\nabla_i a_{lj} + \nabla_j a_{li} - \nabla_l a_{ij}) \\ &= 2h^{kij}\nabla_i H_j \end{aligned} \tag{38}$$

where we used the Codazzi equation in the last step. Then (38) and (15) give

$$\begin{aligned} \frac{\partial}{\partial t}\Delta f &= \frac{\partial}{\partial t}(g^{ij}\nabla_i\nabla_j f) = 2a^{ij}\nabla_i\nabla_j f + \Delta(\frac{\partial}{\partial t}f) - g^{ij}\frac{\partial}{\partial t}(\Gamma_{ij}^k)\nabla_k f \\ &= \Delta(\frac{\partial}{\partial t}f) + 2a^{ij}\nabla_i\nabla_j f + 2h^{kij}\nabla_i H_j\nabla_k f \end{aligned}$$

□

Corollary 3.4 *Let ϕ be a solution of (**) with $d\phi = H$. i.e. ϕ is the mean curvature potential of an exact solution. Then we have*

$$(\frac{\partial}{\partial t} - \Delta)\frac{\partial}{\partial t}\phi = 4a^{ij}\nabla_i\nabla_j\phi \tag{39}$$

We want to derive a Harnack inequality for ϕ. To this end let us define the function

$$h := \Delta\phi - p\frac{|\nabla\phi|^2}{\phi} + q\phi$$

with two positive functions p and q depending only on t and to be determined later.

Lemma 3.5 *Assume that ϕ is a solution of (**) with $d\phi = H$ and $\min_{L_0}\phi = 1$ and that $b_{ij}V^iV^j \leq B|V|^2$ for any $V \in TL$. Then*

$$\begin{aligned} \frac{\partial}{\partial t}h &\geq \Delta h + \frac{p}{n\phi}(h^2 + 2h(\frac{p-2}{\phi}|\nabla\phi|^2 - q\phi)) \\ &+ p(\frac{(p-2)^2}{n} - 2)\frac{|\nabla\phi|^4}{\phi^3} \\ &- (\frac{2p(p-2)q}{n} + \frac{\partial}{\partial t}p + 2B(p + 2\frac{\phi^2}{p}))\frac{|\nabla\phi|^2}{\phi} \\ &+ (\frac{pq^2}{n} + \frac{\partial}{\partial t}q)\phi \end{aligned} \tag{41}$$

Proof: We need the evolution equation for h. With (32) and (39) we get

$$\begin{aligned}
\frac{\partial}{\partial t}h &= \Delta(\frac{\partial}{\partial t}\phi) + 4a^{ij}\nabla_i\nabla_j\phi - (\frac{\partial}{\partial t}p)\frac{|\nabla\phi|^2}{\phi} + (\frac{\partial}{\partial t}q)\phi \\
&- p(\Delta\frac{|\nabla\phi|^2}{\phi} - \frac{2}{\phi}|\nabla_i\nabla_j\phi - \frac{\nabla_i\phi\nabla_j\phi}{\phi}|^2 + \frac{2}{\phi}b_{ij}\nabla^i\phi\nabla^j\phi) + q\Delta\phi \\
&= \Delta h + 4a^{ij}\nabla_i\nabla_j\phi + 2\frac{p}{\phi}|\nabla_i\nabla_j\phi - \frac{\nabla_i\phi\nabla_j\phi}{\phi}|^2 \\
&- (\frac{\partial}{\partial t}p)\frac{|\nabla\phi|^2}{\phi} + (\frac{\partial}{\partial t}q)\phi - \frac{2p}{\phi}b_{ij}\nabla^i\phi\nabla^j\phi
\end{aligned}$$

Using Schwarz' inequality we obtain

$$\begin{aligned}
4a^{ij}\nabla_i\nabla_j\phi &\geq -\frac{2}{\epsilon}|a_{ij}|^2 - 2\epsilon|\nabla_i\nabla_j\phi|^2 \\
&= -\frac{4\phi}{p}b_{ij}\nabla^i\phi\nabla^j\phi - \frac{p}{\phi}|\nabla_i\nabla_j\phi|^2
\end{aligned}$$

where we set $\epsilon = \frac{p}{2\phi}$ and used that $|a_{ij}|^2 = b_{ij}\nabla^i\phi\nabla^j\phi$. With this inequality we can estimate

$$\begin{aligned}
\frac{\partial}{\partial t}h &\geq \Delta h + \frac{p}{\phi}|\nabla_i\nabla_j\phi - 2\frac{\nabla_i\phi\nabla_j\phi}{\phi}|^2 - 2\frac{p}{\phi^3}|\nabla\phi|^4 \\
&- (\frac{\partial}{\partial t}p)\frac{|\nabla\phi|^2}{\phi} + (\frac{\partial}{\partial t}q)\phi - 2(\frac{p}{\phi} + 2\frac{\phi}{p})b_{ij}\nabla^i\phi\nabla^j\phi
\end{aligned}$$

We also have

$$\begin{aligned}
|\nabla_i\nabla_j\phi - 2\frac{\nabla_i\phi\nabla_j\phi}{\phi}|^2 &\geq \frac{1}{n}(\mathrm{trace}(\nabla_i\nabla_j\phi - 2\frac{\nabla_i\phi\nabla_j\phi}{\phi}))^2 \\
&= \frac{1}{n}(\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi})^2 \\
&= \frac{1}{n}(h^2 + 2h(\frac{p-2}{\phi}|\nabla\phi|^2 - q\phi)) \\
&+ \frac{1}{n}(\frac{p-2}{\phi}|\nabla\phi|^2 - q\phi)^2
\end{aligned}$$

With this inequality and the assumption on b_{ij} we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}h &\geq \Delta h + \frac{p}{n\phi}(h^2 + 2h(\frac{p-2}{\phi}|\nabla\phi|^2 - q\phi)) \\
&+ p(\frac{(p-2)^2}{n} - 2)\frac{|\nabla\phi|^4}{\phi^3} \\
&- (\frac{2p(p-2)q}{n} + \frac{\partial}{\partial t}p + 2B(p + 2\frac{\phi^2}{p}))\frac{|\nabla\phi|^2}{\phi} \\
&+ (\frac{pq^2}{n} + \frac{\partial}{\partial t}q)\phi
\end{aligned}$$

which is (41). ∎

Theorem 3.6 *Let ϕ be a solution of (**) with $d\phi = H$, $\min_{L_0}\phi = 1$ and with $\max_{L_0}\phi \le A$. Assume that $b_{ij}V^iV^j \le B|V|^2$ on a time interval $[0,T)$ for any $V \in TL$. Set $k := \frac{2A^2}{1+n}$ and $T_0 := \frac{1}{2(n+1)B}\ln\frac{k+2}{k+1}$. Then the following inequality holds on $[0,\min\{T,T_0\})$*

$$\Delta\phi - p\frac{|\nabla\phi|^2}{\phi} + \frac{n}{tp}\phi \ge 0 \tag{42}$$

where $p = (2+k)e^{-2(n+1)Bt} - k$.

Proof: We use (41) with $q = \frac{n}{tp}$. This gives

$$\begin{aligned}
\frac{\partial}{\partial t}h &\ge \Delta h + \frac{p}{n\phi}(h^2 + 2h(\frac{p-2}{\phi}|\nabla\phi|^2 - \frac{n}{tp}\phi)) \\
&+ p(\frac{(p-2)^2}{n} - 2)\frac{|\nabla\phi|^4}{\phi^3} \\
&- (\frac{\partial}{\partial t}p + 2B(p + 2\frac{\phi^2}{p}))\frac{|\nabla\phi|^2}{\phi} \\
&- \frac{n}{tp^2}(\frac{\partial}{\partial t}p)\phi
\end{aligned}$$

On $[0,\min\{T,T_0\})$ we have

$$1 \le p \le 2$$

and therefore $-\frac{2p(p-2)q}{n} \ge 0$. We also have $\frac{\partial}{\partial t}p = -2(n+1)B(p+k) \le 0$ and the maximum principle and (**) imply that $1 \le \phi \le A$ on $[0,T)$. Consequently

$$\begin{aligned}
p(\frac{(p-2)^2}{n} - 2)\frac{|\nabla\phi|^4}{\phi^3} &\ge -2p\frac{|\nabla\phi|^4}{\phi^3} \ge -2p\frac{|\nabla\phi|^4}{\phi} \\
-(\frac{\partial}{\partial t}p + 2B(p + 2\frac{\phi^2}{p}))\frac{|\nabla\phi|^2}{\phi} &\ge -(-2nBp - 2k(n+1)B + 4BA^2)\frac{|\nabla\phi|^2}{\phi} \\
-\frac{n}{tp^2}(\frac{\partial}{\partial t}p)\phi &\ge 0
\end{aligned}$$

In view of

$$B|\nabla\phi|^2 \ge b_{ij}\nabla^i\phi\nabla^j\phi = |a_{ij}|^2 \ge \frac{1}{n}(\mathrm{trace}(a_{ij}))^2 = \frac{1}{n}|\nabla\phi|^4$$

we also obtain $|\nabla\phi|^4 \le nB|\nabla\phi|^2$ and then

$$\begin{aligned}
\frac{\partial}{\partial t}h &\ge \Delta h + \frac{p}{n\phi}(h^2 + 2h(\frac{p-2}{\phi}|\nabla\phi|^2 - \frac{n}{tp}\phi)) \\
&- (-2nBp - 2k(n+1)B + 4BA^2 + 2nBp)\frac{|\nabla\phi|^2}{\phi} \\
&= \Delta h + \frac{p}{n\phi}(h^2 + 2h(\frac{p-2}{\phi}|\nabla\phi|^2 - \frac{n}{tp}\phi))
\end{aligned}$$

The theorem now follows from the parabolic maximum principle. □

We come to the integral version of the Harnack inequality

Corollary 3.7 *Assume that the assumptions in Theorem (3.6) are satisfied. Further assume that X_1, X_2 are two points on L^n and let $d(X_1, X_2)$ denote the geodesic distance of X_1 to X_2 on (L^n, g_{t_1}). Then*

$$\ln \frac{\phi(X_2,t_2)}{\phi(X_1,t_1)} \geq -n \ln \frac{t_2}{t_1} - e^{2\sqrt{n}B(t_2-t_1)} \frac{(n+1)B}{2(r(t_1)-r(t_2))} d(X_1,X_2)^2 \tag{45}$$

where $r(t) := p(t) + 2(n+1)Bkt$ and $0 < t_1 < t_2 < \min\{T, T_0\}$.

Proof: Let us choose a path $X(t)$ with $X(t_i) = X_i; i = 1, 2$ and set

$$l(t) := \ln(\phi(X(t), t))$$

We obtain

$$\begin{aligned} l(t_2) - l(t_1) &= \int_{t_1}^{t_2} \frac{\partial l}{\partial t} + \langle \nabla l, \dot{X} \rangle dt \\ &= \int_{t_1}^{t_2} \frac{\Delta \phi}{\phi} + \langle \nabla l, \dot{X} \rangle dt \end{aligned}$$

Now we use inequality (42) and Cauchy-Schwarz to estimate

$$l(t_2) - l(t_1) \geq -n \int_{t_1}^{t_2} \frac{1}{tp} dt - \frac{1}{4} \int_{t_1}^{t_2} \frac{1}{p} |\dot{X}|^2_{g_t} dt$$

Let λ denote any eigenvalue of a_{ij}. We certainly have $\lambda^2 \leq |a_{ij}|^2 = b_{ij} \nabla^i \phi \nabla^j \phi$. On the other hand we obtain from the assumptions on b_{ij} that $b_{ij} \nabla^i \phi \nabla^j \phi \leq B|\nabla \phi|^2$. Since $|a_{ij}|^2 \geq \frac{1}{n}(\text{trace}(a_{ij}))^2 = \frac{1}{n}|\nabla \phi|^4$ we obtain $\lambda^2 \leq nB^2$ and $\frac{\partial}{\partial t} g_{ij} = -2a_{ij}$ implies

$$\frac{\partial}{\partial t} g_{ij} \leq 2\sqrt{n} B g_{ij}$$

and then

$$|\dot{X}|^2_{g_t} \leq e^{2\sqrt{n}B(t-t_1)} |\dot{X}|^2_{g_{t_1}}$$

If we assume that X is a geodesic on (L^n, g_{t_1}) parametrized by arclength

$$s = a(r(t_1) - r(t))$$

then we can proceed in the same way as in [3] to compute

$$\int_{t_1}^{t_2} \frac{1}{p} |\dot{X}|^2_{g_t} dt \leq e^{2\sqrt{n}B(t_2-t_1)} \frac{2(n+1)B}{r(t_1)-r(t_2)} d(X_1,X_2)^2$$

In addition we have $p \geq 1$ and therefore

$$\int_{t_1}^{t_2} \frac{1}{tp} dt \leq \ln \frac{t_2}{t_1}$$

This completes the proof. □

Remark: *As one can easily see from the proof of Lemma 2.4 any closed* $1-$*form* η *that evolves according to*

$$\frac{\partial}{\partial t}\eta = dd^\dagger\eta$$

gives rise to a local solution of (∗∗), which is given by the potential of η. *Now let* $W \in \mathbb{R}^{2n}$ *be a fixed vector. We define the following* $1-$*forms on* L

$$\begin{aligned} \eta_i &:= \langle F, \nu_i\rangle \\ \tau_i &:= \langle F, e_i\rangle \\ \lambda_i &:= \langle W, \nu_i\rangle \\ \rho_i &:= \langle W, e_i\rangle \end{aligned}$$

Using equations (8) and (9) we easily deduce that all these forms are closed $(\tau, \lambda, \rho$ *are even exact). The evolution equation for* F *is just*

$$\frac{\partial}{\partial t}F = \Delta F$$

since the mean curvature vector is equal to the Laplacian of F. *This implies the evolution equations*

$$(\frac{\partial}{\partial t} - dd^\dagger)\gamma = \begin{cases} -2H & \text{if } \gamma = \eta \\ 0 & \text{if } \gamma = \tau, \lambda, \rho \end{cases} \tag{55}$$

and in particular $(\frac{\partial}{\partial t} - dd^\dagger)(\eta + 2tH) = 0$. *We also have*

$$\frac{\partial}{\partial t}(|F|^2 + 2nt) = \Delta(|F|^2 + 2nt)$$

in fact $\frac{|F|^2}{2} + nt$ *is just the potential of* τ.

In [5] a matrix Harnack inequality for the mean curvature has been proved for weakly convex hypersurfaces. It would be interesting to find an analogue result for the Lagrangian mean curvature flow. We tried this for a long time. The equations are indeed very similar and appear in a beautiful manner. However the situation for Lagrangian immersions is much harder since the mean curvature flow in this case is rather a coupled system of parabolic equations and therefore the first problem is to find some sort of convexity assumption (or perhaps a totally different assumption) that is preserved during the flow and which would give a good control on the second fundamental form.

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