Nonexistence of minimal Lagrangian spheres in hyperKähler manifolds

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Abstract

We prove that for $n > 1$ one cannot immerse S^{2n} as a minimal Lagrangian manifold into a hyperKähler manifold. More generally we show that any minimal Lagrangian immersion of an orientable closed manifold L^{2n} into a hyperKähler manifold H^{4n} must have nonvanishing second Betti number β_2 and that if $\beta_2 = 1$, L^{2n} is a Kähler manifold and more precisely a Kähler submanifold in H^{4n} w.r.t. one of the complex structures on H^{4n}. In addition we derive a result for the other Betti numbers.

1 Lagrangian immersions in hyperKähler manifolds

$\mathbb{R}^4$ is the simplest example of a hyperKähler manifold. Recall that a hyperKähler manifold $(H^{4n}, \overline{g})$ is a real $4n$-dimensional Riemannian manifold that admits three distinct parallel complex structures $\overline{J}, \overline{K}, \overline{L}$ that are compatible with $\overline{g}$ and that satisfy the relations

$$\overline{J}\,\overline{K} = \overline{L}, \quad \overline{K}\,\overline{L} = \overline{J}, \quad \overline{L}\,\overline{J} = \overline{K}, \tag{1}$$

$$\{\overline{J}, \overline{K}\} = \{\overline{K}, \overline{L}\} = \{\overline{L}, \overline{J}\} = 0 \tag{2}$$

$$\overline{J}^2 = \overline{K}^2 = \overline{L}^2 = -\mathrm{Id}. \tag{3}$$

Throughout this article we will assume that L^{2n} is a Lagrangian submanifold in H^{4n} w.r.t. the complex structure $\overline{J}$, i.e. $J_{kl} = 0$, where for local coordinates $(y^\alpha)_{\alpha=1,\dots,4n}$ on H^{4n} and $(x^k)_{k=1,\dots,2n}$ on L^{2n} we set

$$\overline{J}_{\alpha\beta} := \langle \overline{J}\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\rangle$$

and $J = J_{kl} dx^k \otimes dx^l = \overline{J}_{|TL^{2n}}$. By the Lagrangian condition $\overline{J}$ provides an isomorphism between the tangent and normal spaces at each point on L^{2n}. Consequently one can define the second fundamental form of a Lagrangian immersion in terms of its tangent bundle only, i.e.

$$h(u, v, w) := -\langle \overline{J}u, \overline{\nabla}_v w \rangle.$$

From the Kähler condition w.r.t. $\overline{J}$ one obtains that the second fundamental form is a fully symmetric trilinear tensor on L^{2n}. We will set $h = h_{ijk} dx^i \otimes dx^j \otimes dx^k$. If we take the trace of the second fundamental form w.r.t. the induced metric $g = \overline{g}_{TL}$ we obtain a 1-form $H = H_i dx^i$ called the mean curvature form. Since a hyperKähler manifold is Ricci flat one concludes from the Codazzi equations that H is a closed form. It vanishes if and only if L^{2n} is a minimal Lagrangian immersion. Minimal Lagrangian submanifolds are of great importance in the Riemannian geometry of symplectic manifolds. Only partial results about their existence are known. For an excellent survey article consult [15]. The local existence of special Lagrangian submanifolds (introduced by Harvey and Lawson in [9]) has been proven by Bryant [2]. Since the Lagrangian condition itself is very restrictive on the topology it is expected that the extra condition of being minimal will give further obstructions. The aim of this paper is to demonstrate this for minimal Lagrangian immersions in hyperKähler manifolds. In the sequel we will not distinguish between the 2-forms $\overline{J}_{\alpha\beta}, \overline{K}_{\alpha\beta}, \overline{L}_{\alpha\beta}$ and the corresponding endomorphisms $\overline{J}_\alpha{}^\beta, \overline{K}_\alpha{}^\beta, \overline{L}_\alpha{}^\beta$. An underlined index will denote the application of the complex structure $\overline{J}$, e.g. $\overline{K}_{\underline{i}j}$ is shorthand for $\overline{J}_i{}^\alpha \overline{K}_{\alpha j}$.

Lemma 1.1 *Assume that L^{2n} is Lagrangian w.r.t. $\overline{J}$ and set $K := (\overline{K}_{|TL^{2n}})^T$, $L := (\overline{L}_{|TL^{2n}})^T$. Then we do have the following relations:*

$$K^2 + L^2 = -Id, \tag{6}$$

$$[K, L] = 0. \tag{7}$$

Proof: Let $e_1, \dots, e_{2n}$ be an orthonormal frame. Set $\nu_k := \overline{J}e_k$. A tangent vector V on L^{2n} can be written as

$$V = \langle V, e_i \rangle e_i,$$

where doubled latin indices are summed from 1 to $2n$. Then

$$\begin{aligned} KV &= \overline{K}V - (\overline{K}V)^\perp \\ &= \langle V, e_i \rangle \overline{K}e_i - \langle V, e_i \rangle \langle \overline{K}e_i, \nu_k \rangle \nu_k \\ &= \langle V, e_i \rangle \overline{K}e_i - \langle V, e_i \rangle \langle e_i, \overline{L}e_k \rangle \nu_k \\ &= \langle V, e_i \rangle \overline{K}e_i - \langle V, \overline{L}e_k \rangle \nu_k \\ &= \langle V, e_i \rangle \overline{K}e_i + \langle \overline{L}V, e_k \rangle \nu_k \\ &= \langle V, e_i \rangle \overline{K}e_i + \langle LV, e_k \rangle \nu_k. \end{aligned} \tag{9}$$

From this we obtain

$$\begin{aligned}\overline{K}KV &= \langle V, e_i\rangle \overline{K}^2 e_i + \langle LV, e_k\rangle \overline{KJ} e_k \\ &= -\langle V, e_i\rangle e_i - \langle LV, e_k\rangle \overline{L} e_k \\ &= -V - \langle LV, e_k\rangle \overline{L} e_k\end{aligned}$$

and then

$$\begin{aligned}K^2V &= (\overline{K}KV)^T \\ &= -V - \langle LV, e_k\rangle\langle \overline{L}e_k, e_l\rangle e_l \\ &= -V - \langle \overline{L}LV, e_l\rangle e_l \\ &= -V - \langle L^2V, e_l\rangle e_l \\ &= -V - L^2V.\end{aligned}$$

That proves (6). Applying $\overline{L}$ to equation (9) we get

$$\begin{aligned}\overline{L}KV &= \langle V, e_i\rangle \overline{LK} e_i + \langle LV, e_k\rangle \overline{LJ} e_k \\ &= -\langle V, e_i\rangle \nu_i + \langle LV, e_k\rangle \overline{K} e_k\end{aligned}$$

hence

$$\begin{aligned}LKV &= \langle LV, e_k\rangle\langle \overline{K}e_k, e_l\rangle e_l \\ &= \langle \overline{K}LV, e_l\rangle e_l \\ &= \langle KLV, e_l\rangle e_l = KLV.\end{aligned}$$

Therefore K and L commute. □

2 The result

Definition 2.1 *Let $\overline{\eta}$ be a p-form on a Hermitian manifold $(H, \overline{J}, \overline{g})$. $\overline{\eta}$ is called* **anti-Hermitian** *if for any vectors $V_1, \ldots, V_p$ we have*

$$\overline{\eta}(JV_1, V_2, V_3, \ldots, V_p) = \overline{\eta}(V_1, JV_2, V_3, \ldots, V_p). \tag{10}$$

A p-form will be called **covariantly Hermitian** *if for any vectors $W, V_1, \ldots, V_p$ we have*

$$\overline{\nabla}_{JW}\overline{\eta}(V_1, V_2, \ldots, V_p) = -\overline{\nabla}_W\overline{\eta}(JV_1, V_2, \ldots, V_p). \tag{11}$$

Remark: *If $\overline{\eta}$ is anti-Hermitian then we conclude that*

$$\overline{\eta}(V_1, \ldots, V_{i-1}, JV_i, V_{i+1}, \ldots, V_p) = \overline{\eta}(V_1, \ldots, V_{j-1}, JV_j, V_{j+1}, \ldots, V_p) \tag{12}$$

for any i, j.

Let us define the set

$$O^p(H) := \{\overline{\eta} | \overline{\eta} \text{ is a covariantly Hermitian and anti-Hermitian } p\text{-form on } H\}.$$

Note that a parallel p-form on H is covariantly Hermitian.

Lemma 2.2 *Assume that L is a minimal Lagrangian immersion into a Kähler manifold $(H, \overline{J}, \overline{g})$ and that $\overline{\eta} \in O^p$ is harmonic. Then $\eta := \overline{\eta}_{|TL}$ is a (possibly trivial) harmonic p-form on L.*

Proof: With the Gauß-Weingarten equations we compute

$$\nabla_i \eta_{j_1 \cdots j_p} = \overline{\nabla}_i \overline{\eta}_{j_1 \cdots j_p} - h^n{}_{ij_1} \overline{\eta}_{\underline{n} j_2 \cdots j_p} - \cdots - h^n{}_{ij_p} \overline{\eta}_{j_1 \cdots j_{p-1} \underline{n}}.$$

and then

$$\begin{aligned}
(\delta\eta)_{j_2 \ldots j_p} &= -\nabla^i \eta_{ij_2 \cdots j_p} \\
&= -\overline{\nabla}^i \overline{\eta}_{ij_2 \cdots j_p} + H^n \overline{\eta}_{\underline{n} j_2 \cdots j_p} + \sum_{l=2}^{p} h^{ni}{}_{j_l} \overline{\eta}_{ij_2 \ldots j_{l-1} \underline{n} j_{l+1} \ldots j_p}
\end{aligned}$$

and since $\overline{\eta}$ is anti-Hermitian we conclude that $\overline{\eta}_{ij_2 \ldots j_{l-1} \underline{n} j_{l+1} \cdots j_p}$ is anti symmetric in i and n and since therefore

$$\sum_{l=2}^{p} h^{ni}{}_{j_l} \overline{\eta}_{ij_2 \ldots j_{l-1} \underline{n} j_{l+1} \cdots j_p}$$

is the contraction of an anti symmetric tensor with a symmetric tensor and the minimality of L implies that $H^n = 0$ we see that the last equation reduces to

$$(\delta\eta)_{j_2 \ldots j_p} = -\overline{\nabla}^i \overline{\eta}_{ij_2 \cdots j_p}.$$

By the Lagrangian condition we have

$$(\overline{\delta}\overline{\eta})_{j_2 \cdots j_p} = -g^{ij} (\overline{\nabla}_i \overline{\eta}_{jj_2 \cdots j_p} + \overline{\nabla}_{\underline{i}} \overline{\eta}_{\underline{j} j_2 \cdots j_p}).$$

and since $\overline{\eta}$ is covariantly Hermitian and $\overline{J}^2 = -\mathrm{Id}$ we see that

$$\frac{1}{2} (\overline{\delta}\overline{\eta})_{j_2 \ldots j_p} = -\overline{\nabla}^i \overline{\eta}_{ij_2 \cdots j_p}.$$

Since η is the restriction of a closed form $\overline{\eta}$ we see that η is both closed and coclosed hence a harmonic p-form. Thus we obtain that the restriction of any harmonic form $\overline{\eta} \in O^p(H)$ to a minimal Lagrangian submanifold becomes harmonic w.r.t. the induced metric on L.

Corollary 2.3 *If the restriction of a harmonic p-form $\overline{\eta} \in O^p(H)$ to a minimal Lagrangian submanifold is nonzero at one point on L then the p-th Betti number $\beta_p(L)$ of L is positive.*

□

Theorem 2.4 *Let L^{2n} be a compact, orientable, minimal Lagrangian immersion into a hyperKähler manifold H^{4n}. Then L^{2n} is either Kähler or the second Betti number β_2 of L^{2n} satisfies $\beta_2 \geq 2$.*

Remark: *Note that in both cases we have $\beta_2 \geq 1$.*

Proof: First we compute

$$\nabla_i K_{jk} = \overline{\nabla}_i \overline{K}_{jk} - h_{ij}{}^n \overline{K}_{\underline{n}k} - h_{ik}{}^n \overline{K}_{j\underline{n}}$$

and since

$$\overline{K}_{\underline{n}k} = \overline{J}_n{}^\alpha \overline{K}_{\alpha k} = L_{kn}$$

and $\overline{\nabla K} = 0$ we obtain

$$\nabla_i K_{jk} = h_{ij}{}^n L_{nk} - h_{ik}{}^n L_{nj}. \tag{21}$$

Similarly

$$\overline{L}_{\underline{n}k} = \overline{J}_n{}^\alpha \overline{L}_{\alpha k} = K_{nk}$$

and

$$\nabla_i L_{jk} = -h_{ij}{}^n K_{nk} + h_{ik}{}^n K_{nj}. \tag{23}$$

$\overline{K}$ and $\overline{L}$ are both harmonic forms in $O^2(H)$. Then Lemma 2.2 implies that K and L are harmonic 2-forms on L. If K and L represent two independent elements in $H^2(L^{2n})$ we are done. Otherwise there exist two constants s and c such that

$$cK - sL = 0, \; c^2 + s^2 = 1.$$

Let s be nonzero (if c is nonzero we can proceed similarly). Then equations (21) and (23) prove that for $C := sK + cL$ we must have

$$\nabla_i C_{jk} = h_{ij}{}^n (sL_{nk} - cK_{nk}) - h_{ik}{}^n (sL_{nj} - cK_{nj}) = 0$$

and

$$C^2 = s^2 K^2 + sc\{K, L\} + c^2 L^2 = (s^2 + 2c^2 + s^{-2} c^4) K^2 = s^{-2} K^2.$$

Since by Lemma 1.1 we have $K^2 + L^2 = -\mathrm{Id}$ and $L = s^{-1} cK$ we conclude that $s^{-2} K^2 = -\mathrm{Id}$ and consequently C is a parallel complex structure on L^{2n}. This proves that (L^{2n}, C, g) must be Kähler. □

Corollary 2.5 *If L^{2n} is a compact, orientable, minimal Lagrangian immersion in a hyperKähler manifold H^{4n} and the second Betti number of L^{2n} equals 1, then L^{2n} is a Kähler submanifold w.r.t. one of the complex structures on H^{4n}.*

Corollary 2.6 *If $n > 1$ then we cannot immerse S^{2n} as a minimal Lagrangian sphere into a hyperKähler manifold.*

Remark: *If we drop the minimality condition then this is wrong since the Whitney immersions of S^{2n} in $\mathbb{R}^{4n}$ given by the restriction of the map*

$$\begin{aligned} f &: \mathbb{R}^{2n+1} \to \mathbb{C}^{2n} \\ f(x^0, \ldots, x^{2n}) &:= \frac{1}{1+(x^0)^2}(x^1, \ldots, x^{2n}, x^0x^1, \ldots, x^0x^{2n}) \end{aligned}$$

to $S^{2n} \subset \mathbb{R}^{2n+1}$ provide counterexamples. If we drop the hyperKähler condition then Theorem 2.4 is no longer true since we can then take the double covering of $\mathbb{RP}^n$ by S^n and use the fact that the natural inclusion $\mathbb{RP}^n \subset \mathbb{CP}^n$ gives a minimal Lagrangian immersion of S^n into $\mathbb{CP}^n$. In addition T^*S^2 with the complex structure of an affine quadric and equipped with the Eguchi-Hanson metric is a hyperKähler manifold for which the zero section S^2 is a minimal Lagrangian immersion.

More generally we have

Corollary 2.7 *If L^{2n} is a compact, orientable manifold such that $\beta_2 \leq 1$ and either one of the even Betti numbers vanishes or one of the odd Betti numbers is odd. Then we cannot find a minimal Lagrangian immersion of L^{2n} into a hyperKähler manifold H^{4n}.*

Proof: This follows from the fact that the even Betti numbers of a Kähler manifold are positive and the odd Betti numbers are even. □

Corollary 2.8 *If L^{2n} is a compact, minimal Lagrangian immersion in a hyperKähler manifold H^{4n} and the second Betti number of L^{2n} vanishes, then L^{2n} is not orientable.*

The next proposition explains some of the mystery of Theorem 2.4.

Proposition 2.9 *Assume that L^2 is an oriented Lagrangian immersion in a hyperKähler manifold H^4. Then the induced conformal structure C on L^2 can be expressed in terms of K, L and the Lagrangian angle α, i.e.*

$$C = \sin\alpha K + \cos\alpha L, \tag{27}$$

where $d_i\alpha = H_i$.

Proof: Since the dimension of L^2 is two we can rewrite equations (21) and (23) as

$$\begin{aligned} \nabla_i K_{jk} &= H_i L_{jk}, \\ \nabla_i L_{jk} &= -H_i K_{jk}. \end{aligned} \tag{28}$$

Now assume that V is a unit tangent vector to L^2 at some point p. Then $\langle KV, V\rangle = \langle\overline{K}V, V\rangle = 0$ and also $\langle LV, V\rangle = 0$. This proves that at p we can find two constants c and s such that

$$\begin{aligned} K &= sC, \\ L &= cC. \end{aligned}$$

By Lemma 1.1 and from $C^2 = -\mathrm{Id}$ we conclude that $c^2 + s^2 = 1$. This construction depends smoothly on p and consequently we can find a function β such that

$$\begin{aligned} K &= \sin\beta C, \\ L &= \cos\beta C. \end{aligned}$$

But then since C is parallel

$$\begin{aligned} \nabla_i K_{jk} &= \cos\beta \nabla_i \beta C_{jk} = \nabla_i \beta L_{jk}, \\ \nabla_i L_{jk} &= -\sin\beta \nabla_i \beta C_{jk} = -\nabla_i \beta K_{jk}. \end{aligned}$$

Comparing this with equations (28) we conclude that $\alpha - \beta = \text{const}$. □

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