A relation between mean curvature flow solitons and minimal submanifolds

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Abstract. We derive a one to one correspondence between conformal solitons of the mean curvature flow in an ambient space N and minimal submanifolds in a different ambient space $\widetilde{N}$, where $\widetilde{N}$ equals $\mathbb{R} \times N$ equipped with a warped product metric and show that a submanifold in N converges to a conformal soliton under the mean curvature flow in N if and only if its associated submanifold in $\widetilde{N}$ converges to a minimal submanifold under a rescaled mean curvature flow in $\widetilde{N}$. We then define a notion of stability for conformal solitons and obtain L^p estimates as well as pointwise estimates for the curvature of stable solitons.

1. Introduction

Let M be a smooth manifold, (N, g) a smooth Riemannian manifold and assume that for $t \in [0, T)$

$$F_t : M \to N \tag{1.1}$$

is a smooth family of immersions into (N, g). If F_t satisfies the evolution equation

$$\frac{d}{dt} F_t = \overrightarrow{H}_t, \tag{1.2}$$

where $\overrightarrow{H}_t$ denotes the mean curvature vector field of $F_t(M) \subset N$, then F_t is called a solution of the mean curvature flow (MCF) with initial value $F_0(M)$. Minimal surfaces are the stationary solutions of (1.2). There are a couple of interesting solutions of (1.2) namely those solutions that move along the integral curves of a smooth vector field X in N. In the case where X is a conformal vector field they are called conformal solitons. If X is even parallel, then they are better known as translating solitons. Standard examples for solitons are the translating "grim reaper" $y = -\log(\cos(x))$ in $\mathbb{R}^2$ and the self-similarly moving curves studied by Abresch and Langer [1]. In the

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former case $X = \frac{\partial}{\partial y}$ is a fixed vector in R^2 whereas in the latter case $X(x) = -x$ is the conformal vector field describing homotheties. Angenent has shown the existence of self-shrinking doughnuts [2]. Solitons are relevant for the investigation of rescaled MCF singularities [4], [10]. Only partial results about them are known [6], [9], [11], [12]. There has been found recent proof for the existence of Lagrangian MCF solitons as well [17]. The typical difficulty to establish results for solitons is both their great variety and the nonexistence of a uniform theory for their nature. We also mention a well-known remark, probably dating back to Huisken, is that the homothetic shrinking soliton equation in R^{n+1} is the first variation of the measure $e^{h(x)}dH^n$ integrated over a hypersurface M^n. This can be achieved as the area measure of a suitable conformal metric on R^n. Since minimal submanifolds are the critical points for the MCF (which is the negative gradient flow for the volume of a submanifold) but solitons are the more frequent singularities, it is natural to presume a link between these two classes. Surprisingly we can actually find such an identification in the case of conformal solitons. If X is a non-parallel Killing field, i.e. X generates a 1-parameter family of isometries, our method fails due to the non-integrability condition $\nabla_i X_j = -\nabla_j X_i$. One of the most absorbing issues when discussing solitons is the question of their stability. Is it true that small perturbations of a soliton will converge under the MCF (perhaps with some additional rescaling in space-time) to the same soltion? Since we will identify a soliton with a minimal submanifold, this question will reduce to the question whether a perturbation of it will converge to this minimal submanifold or not. In [12] Hungerbühler and the author derived some stability results for rotating solitons in R^n. Here we will focus on the conformal ones. Let F_t be a solution of (1.2) that moves along the integral curves of a vector field X. Then a necessary condition for F_0 is the validity of the following equation:

$$\overrightarrow{H} = X^{\perp}, \tag{1.3}$$

where $\overrightarrow{H}$ is the mean curvature vector field of $F_0(M)$ and $^{\perp}$ denotes the projection onto the normal bundle. However if $F_0(M)$ satisfies (1.3), then this does not automatically imply that the mean curvature flow with initial data $F_0(M)$ moves along the integral curves of X. In the case where X is a Killing field, X generates a 1-parameter group of isometries and (1.3) is adequate. A vector field X is conformal if $\nabla_j X_i + \nabla_i X_j = 2\lambda g_{ij}$ for a function λ. In this paper we restrict our attention to the case where X satisfies

$$\nabla_j X_i = \lambda g_{ij} \tag{1.4}$$

for a smooth function λ. We note that by Obata's theorem [13] there are some obstructions for N admitting conformal vector fields of the form (1.4). The action of parallel vector fields on N is similar to the translations along a fixed vector in R^n (justifying the term "translating soliton").

2. A warped product metric

Let (N, g) be a Riemannian manifold and $f : N \to \mathrm{R}$ a smooth function. The warped product metric $\widetilde{g}$ on $\widetilde{N} := \mathrm{R} \times N$ shall be defined by

$$\widetilde{g}(s, x) := e^{2f(x)}ds^2 + g(x), \tag{2.1}$$

where $x \in N$ and ds^2 is the usual line element on R.

$$\phi_{s_0}(s,x) := (s+s_0,x) \tag{2.2}$$

gives a 1-parameter family of isometries on $\widetilde{N}$ and the projection

$$\begin{aligned} \pi &: \widetilde{N} \to N \\ \pi(s,x) &:= x \end{aligned}$$

becomes a Riemannian submersion $\pi : (\widetilde{N}, \widetilde{g}) \to (N, g)$ with fibers $\pi^{-1}(x) =: [x] = \mathrm{R} \times \{x\}$. We are interested in the geometry of N and $[x]$.

Lemma 2.1. *The mean curvature vector field $\overrightarrow{H}_{[x]}$ of $[x]$ in $\widetilde{N}$ at (s,x) is given by $(0, -\nabla f(x))$, where ∇ is the covariant derivative in N.*

Proof: We define the immersion

$$\begin{aligned} p &: \mathrm{R} \to \widetilde{N} \\ p(s) &:= (s,x). \end{aligned}$$

Therefore $p = [x]$. Let us choose normal coordinates for g around $x \in N$. Since $\frac{\partial p}{\partial s} = (1,0)$ we compute that

$$\nu_i := (0, \frac{\partial}{\partial x^i}), i = 1, \dots, n \tag{2.3}$$

forms an orthonormal basis of $(T_{(s,x)}[p])^\perp$ for any $s \in \mathrm{R}$. Let us label the first coordinate s in $\widetilde{N}$ with a zero and the other n ($n = \dim(N)$) coordinates on N by latin indices i,j,k. Greek indices range from 0 to n. The equation of Gauss-Weingarten is:

$$\frac{\partial^2 p^\alpha}{\partial s^2} - \Gamma^0_{00} \frac{\partial p^\alpha}{\partial s} + \widetilde{\Gamma}^\alpha_{\beta\gamma} \frac{\partial p^\beta}{\partial s} \frac{\partial p^\gamma}{\partial s} = -h^i{}_{00} \nu^\alpha_i, \tag{2.4}$$

where doubled latin indices are summed from 1 to n and $h^i{}_{00}$ is the second fundamental form of $[p]$ w.r.t. the normal vector ν_i. From this we get

$$H_i = -e^{-2f} (\frac{\partial^2 p^\alpha}{\partial s^2} \nu^\beta_i \widetilde{g}_{\alpha\beta} + \widetilde{\Gamma}^\alpha_{\beta\gamma} \frac{\partial p^\beta}{\partial s} \frac{\partial p^\gamma}{\partial s} \nu^\delta_i \widetilde{g}_{\alpha\delta}) \tag{2.5}$$

and then also

$$H_i = -e^{-2f} \widetilde{\Gamma}^i_{00} = \nabla_i f. \tag{2.6}$$

The result now is a consequence of $\overrightarrow{H}_{[x]} = -\sum_{i=1}^n H_i \nu_i$. □

The curvature of $\widetilde{N}$ can be computed easily from that of N (see [14]). We obtain the following relations

Proposition 2.2.

$$\widetilde{g}_{\alpha\beta} = \begin{pmatrix} e^{2f} & 0 \\ 0 & g_{ij} \end{pmatrix}, \quad \widetilde{g}^{\alpha\beta} = \begin{pmatrix} e^{-2f} & 0 \\ 0 & g^{ij} \end{pmatrix}. \tag{2.7}$$

$$\widetilde{R}_{ijkl} = R_{ijkl} \tag{2.8}$$
$$\widetilde{R}_{0ijk} = 0 \tag{2.9}$$
$$\widetilde{R}_{0i0j} = e^{2f}(\nabla_i\nabla_j f - \nabla_i f \nabla_j f). \tag{2.10}$$

The Ricci curvature tensor is given by

$$\widetilde{R}_{\alpha\beta} = \begin{pmatrix} e^{2f}(\Delta f - |\nabla f|^2) & 0 \\ 0 & R_{ij} + (\nabla_i\nabla_j f - \nabla_i f \nabla_j f) \end{pmatrix}. \tag{2.11}$$

We will also need the expression for the covariant derivative of $\widetilde{R}_{\alpha\beta\gamma\delta}$:

Proposition 2.3.

$$\begin{aligned}
\widetilde{\nabla}_0 \widetilde{R}_{ijkl} &= 0 \\
\widetilde{\nabla}_n \widetilde{R}_{ijkl} &= \nabla_n R_{ijkl} \\
\widetilde{\nabla}_0 \widetilde{R}_{0ijk} &= e^{2f}(\nabla^n f R_{nijk} + \nabla_i\nabla_j f \nabla_k f - \nabla_i \nabla_k f \nabla_j f) \\
\widetilde{\nabla}_l \widetilde{R}_{0ijk} &= 0 \\
\widetilde{\nabla}_0 \widetilde{R}_{0i0j} &= 0 \\
\widetilde{\nabla}_k \widetilde{R}_{0i0j} &= e^{2f}(\nabla_k\nabla_i\nabla_j f - \nabla_i f \nabla_k \nabla_j f - \nabla_j f \nabla_k \nabla_i f).
\end{aligned}$$

Now let $M \subset N$ be a hypersurface. Denote by $\widetilde{M}$ the submanifold in $\widetilde{N}$ given by $\widetilde{M} = \mathrm{R} \times N$ and call it the associated submanifold to M. If ν is a normal vector at $x \in M$, then $\widetilde{\nu} := (0, \nu)$ is a normal vector at $(s, x) \in \widetilde{M}$. It is well known (e.g. see [16]) that the mean curvature vector $\overrightarrow{H}_{\widetilde{M}}$ of $\widetilde{M}$ at the point (s, x) is then given by

$$\overrightarrow{H}_{\widetilde{M}}(s,x) = (0, \overrightarrow{H}_M(x)) + \overrightarrow{H}^{\perp}_{[x]}, \tag{2.12}$$

where $\overrightarrow{H}_M(x)$ is the mean curvature vector of $M \subset N$ at x and $^\perp$ denotes the projection onto the normal bundle of $\widetilde{M}$ (that can also be identified with the normal bundle of M). From Lemma 2.1 we immediately obtain

Corollary 2.4. *The mean curvature vector of $\widetilde{M}$ at the point (s, x) is given by*

$$\overrightarrow{H}_{\widetilde{M}}(s,x) = \big(0, \overrightarrow{H}_M(x) - (\nabla f(x))^{\perp}\big). \tag{2.13}$$

3. Minimality and stability

Theorem 3.1. *Assume that X is a conformal vector field on a simply connected Riemannian manifold (N, g) satisfying (1.4). Then there exists a warped product metric $\widetilde{g}$ on $\widetilde{N} = \mathrm{R} \times N$ such that a submanifold $M \subset N$ satisfies the soliton equation (1.3)* *if and only if the associated submanifold* $\widetilde{M} = \mathrm{R} \times M \subset \widetilde{N}$ *is a minimal submanifold in* $(\widetilde{N}, \widetilde{g})$.

Proof: Since $\nabla_k X_j$ is symmetric in k and j we can integrate X_i locally, i.e. on any simply connected domain $\Omega \subset N$ we can find a smooth function f such that $X_i = \nabla_i f$. The theorem then follows from Corollary 2.4 if we choose $\widetilde{g}(s,x) = e^{2f(x)}ds^2 + g(x)$.

□

Theorem 3.1 enables us to obtain numerous results for conformal solitons. In fact any general result for minimal submanifolds can be transformed to an analog statement for MCF solitons. The first thing which comes into mind is the stability criterion for minimal submanifolds. Recall that a minimal hypersurface $\widetilde{M} \subset \widetilde{N}$ is called stable if and only if

$$\int_{\widetilde{M}} (|\widetilde{A}|^2 + \widetilde{\mathrm{Ric}}(\widetilde{\nu}, \widetilde{\nu})\widetilde{u}^2 d\widetilde{\mu} \leq \int_{\widetilde{M}} |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu} \tag{3.1}$$

for all $\widetilde{u} \in C_0^\infty(\widetilde{M})$. The question now is: “What is the analog stability criterion for conformal solitons”? To this end assume that $M \subset N$ is a hypersurface. Denote the sum of the squared principle curvatures of M by $|A|^2$ and similarly $|\widetilde{A}|^2$ is the squared norm of the second fundamental form for $\widetilde{M} \subset \widetilde{N}$. Any tensor T on M (resp. on N) can be extended to a tensor $\widetilde{T}$ on $\widetilde{M}$ (resp. on $\widetilde{N}$).

Lemma 3.2. *We have*

$$|\widetilde{A}|^2 + \widetilde{\mathrm{Ric}}(\widetilde{\nu}, \widetilde{\nu}) = |A|^2 + \mathrm{Ric}(\nu, \nu) + \nabla_\nu \nabla_\nu f. \tag{3.2}$$

Proof: Since $\widetilde{\nu} = (0, \nu)$ we see from (2.11) that

$$\widetilde{\mathrm{Ric}}(\widetilde{\nu}, \widetilde{\nu}) = \mathrm{Ric}(\nu, \nu) + \nabla_\nu \nabla_\nu f - \langle \nu, \nabla f \rangle^2, \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on TN. From [16] we have that

$$|\widetilde{A}|^2 = |A|^2 + \kappa^2, \tag{3.4}$$

where $\kappa = -\widetilde{g}(\widetilde{\nu}, \overrightarrow{H}_{[x]})$ is the principle curvature of $\widetilde{M}$ that belongs to the fiber direction. But then Lemma 2.1 gives the result.

□

Definition 3.3. If $\widetilde{M}$ is the associated submanifold to M, then we call a deformation of $\widetilde{M}$ symmetric, if it is constant along the fiber directions $[x]$. A minimal submanifold $\widetilde{M}$ in $\widetilde{N}$ that is associated to a submanifold $M \subset N$ is called symmetric stable if

$$\int_{[0,1]\times M} (|\widetilde{A}|^2 + \widetilde{\mathrm{Ric}}(\widetilde{\nu}, \widetilde{\nu}))\widetilde{u}^2 d\widetilde{\mu}(s,x) \leq \int_{[0,1]\times M} |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu}(s,x) \tag{3.5}$$

for all $\widetilde{u} \in C^\infty_{0,\mathrm{sym}}([0,1] \times M) := \{\widetilde{u} : \widetilde{u}(s,x) = \widetilde{u}(0,x)\ \forall s \in [0,1]$ and $u(x) := \widetilde{u}(0,x) \in C_0^\infty(M)\}$. Here $\widetilde{D}$ denotes the induced connection on $\widetilde{M}$.

Remark 3.4. Thus a minimal surface is symmetric stable if it is stable under all symmetric deformations $u \in C^\infty_{0,\mathrm{sym}}([0,1]\times M)$. Note that $C^\infty_{0,\mathrm{sym}}([0,1]\times M)$ is not a subset of $C^\infty_0([0,1]\times M)$ but that for any smooth function f on M we obtain an identification of $C^\infty_0(M)$ with $C^\infty_{0,\mathrm{sym}}([0,1]\times M)$ via $u\mapsto \widetilde{u} := ue^{-\frac{f}{2}}$.

Lemma 3.5. *Let X be a conformal vector field on a simply connected manifold N, s.t. $\nabla_i X_j = \lambda g_{ij}$ for a smooth function λ. Further assume that $M\subset N$ is a hypersurface that solves the soliton equation (1.3). Then there exists a smooth function f on N with $\nabla f = X$ (uniquely defined up to adding a constant) such that the associated minimal hypersurface $\widetilde{M}\subset(\widetilde{N}, e^{2f}ds^2+g)$ is stable under symmetric deformations if and only if*

$$\int_M (|A|^2 + \mathrm{Ric}(\nu,\nu) - \frac{1}{4}H^2 - \frac{m-2}{2}\lambda - \frac{1}{4}|X|^2)u^2 d\mu \le \int_M |Du|^2 d\mu \tag{3.6}$$

for any test function $u\in C^\infty_0(M)$ (Here $d\mu$ denotes the volume element and m the dimension of M).

Proof: Let us use Theorem 3.1. The function f is unique up to a constant since for any two such functions f_1, f_2 we must have $\nabla f_1 = \nabla f_2 = X$. A minimal hypersurface $\widetilde{M}\subset\widetilde{N}$ is symmetric stable if and only if

$$\int_{[0,1]\times M} (|\widetilde{A}|^2 + \widetilde{\mathrm{Ric}}(\widetilde{\nu},\widetilde{\nu}))\widetilde{u}^2 d\widetilde{\mu}(s,x) \le \int_{[0,1]\times M} |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu}(s,x) \tag{3.7}$$

for all $\widetilde{u}\in C^\infty_{0,\mathrm{sym}}([0,1]\times M)$. Since $\nabla f = X$ and $\nabla X = \lambda g$ we conclude that $\nabla_\nu\nabla_\nu f = \lambda g(\nu,\nu) = \lambda$ and then in a first step with Lemma 3.2

$$\begin{aligned}&\int_{[0,1]\times M} (|\widetilde{A}|^2 + \widetilde{\mathrm{Ric}}(\widetilde{\nu},\widetilde{\nu}))\widetilde{u}^2 - |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu}(s,x)\\ &= \int_{[0,1]\times M} (|A|^2 + \mathrm{Ric}(\nu,\nu)+\lambda)\widetilde{u}^2 - |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu}(s,x).\end{aligned}$$

From (2.7) we obtain $d\widetilde{\mu}(s,x) = e^{f(x)}ds\,d\mu(x)$. Let us set $\widetilde{u}(s,x) := u(x)e^{-\frac{f(x)}{2}}$ for a function $u\in C^\infty_0(M)$. Then

$$|\widetilde{D}\widetilde{u}|^2 = e^{-f}(|Du|^2 + \frac{1}{4}u^2|Df|^2 - u\langle Du, Df\rangle), \tag{3.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the induced inner product on M. This gives

$$\begin{aligned}&= \int_{[0,1]\times M} (|A|^2 + \mathrm{Ric}(\nu,\nu)+\lambda)\widetilde{u}^2 - |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu}(s,x)\\ &= \int_{[0,1]}\int_M (|A|^2 + \mathrm{Ric}(\nu,\nu)+\lambda)u^2 - |Du|^2 - \frac{1}{4}u^2|Df|^2 + u\langle Du, Df\rangle d\mu ds\\ &= \int_M (|A|^2 + \mathrm{Ric}(\nu,\nu)+\lambda)u^2 - |Du|^2 - \frac{1}{4}u^2|Df|^2 + u\langle Du, Df\rangle d\mu. \end{aligned} \tag{3.9}$$

Next we observe that for any two vectors $V, W \in TM$ we have

$$D^2 f(V,W) = \nabla^2 f(V,W) - A(V,W)\nabla f(\nu), \tag{3.10}$$

where A denotes the second fundamental form of M in N. Since $\nabla^2 f(V,W) = \lambda g(V,W)$ and $X^\perp = \overrightarrow{H} = -H\nu = \nabla f^\perp$ we conclude that

$$\Delta_D f = m\lambda + H^2. \tag{3.11}$$

A partial integration for $\int u\langle Du, Df\rangle d\mu$ then gives

$$\int_M u\langle Du, Df\rangle d\mu = -\frac{1}{2}\int_M u^2(m\lambda + H^2)d\mu. \tag{3.12}$$

On the other hand

$$|Df|^2 = |\nabla f|^2 - |\nabla f^\perp|^2 = |\nabla f|^2 - H^2 = |X|^2 - H^2. \tag{3.13}$$

Combining equations (3.9), (3.12) and (3.13) we get

$$\begin{aligned} &\int_{[0,1]\times M} (|\widetilde{A}|^2 + \widetilde{\mathrm{Ric}}(\widetilde{\nu},\widetilde{\nu}))\widetilde{u}^2 - |\widetilde{D}\widetilde{u}|^2 d\widetilde{\mu}(s,x) \\ = &\int_M (|A|^2 + \mathrm{Ric}(\nu,\nu) - \frac{1}{4}H^2 - \frac{m-2}{2}\lambda - \frac{1}{4}|X|^2)u^2 - |Du|^2 d\mu. \end{aligned}$$

This proves the Lemma. □

Definition 3.6. In view of Lemma 3.5 we call a conformal soliton stable if it satisfies (3.6).

Remark 3.7. Let $(N,g) = (\mathrm{R}^{m+1},\delta)$ be the euclidean space and assume that $X(x) = -x$ is the conformal radial vector field. Then $f(x) = -\frac{|x|^2}{2}$ is a global generator for X and the conformal solitons for X are self-similarly moving solutions to the MCF. We compute $\nabla_i\nabla_j f = -1\cdot\delta_{ij}$ and thus $\lambda = -1$ and $|X|^2 = r^2$, where r denotes the distance to the origin. If M is a hypersphere of radius r and with outward pointing normal vector ν, then $H = \frac{m}{r}$ and $\langle X,\nu\rangle = -r$. Therefore the soliton equation $\overrightarrow{H} = -H\nu = X^\perp = \langle X,\nu\rangle$ becomes $\frac{m}{r} = r$. Then since $r^2 = m$ it can be easily seen that

$$\begin{aligned} &|A|^2 + \mathrm{Ric}(\nu,\nu) - \frac{1}{4}H^2 - \frac{m-2}{2}\lambda - \frac{1}{4}|X|^2 \\ = &\frac{m}{r^2} + 0 - \frac{m^2}{4r^2} + \frac{m-2}{2} - \frac{r^2}{4} = 0, \end{aligned}$$

i.e we observe that for the self-shrinking spheres in R^{m+1} the second variation of the volume functional for the associated submanifold in R^{m+2} vanishes identically and that these spheres are stable in the sense of Definition 3.6. The sphere of radius $r^2 = m$

is stationary for the flow $\frac{d}{dt}F = -(H - \langle F, \nu\rangle)\nu$ but if we consider a sphere with a different radius r_0 then we see that $-(H - \langle F, \nu\rangle)\nu$ is outward pointing for $r_0 > r$ and inward pointing for $r_0 < r$. This means that spheres of different radii will not converge to the stationary sphere of radius $r = \sqrt{m}$ under the rescaled mean curvature flow $\frac{d}{dt}F = -(H - \langle F, \nu\rangle)\nu$. Of course the preceding considerations just show that the L^2-derivatives of the volume form up to second order can vanish although the submanifold is not area minimizing.

4. L^p estimate

Throughout the rest of the paper we want to assume that the sectional curvatures of (N, g) are bounded between two constants, i.e.

$$K_2 \leq \sigma \leq K_1, \tag{4.1}$$

where σ denotes any sectional curvature of (N, g) and we will also assume

$$|\nabla R|^2 \leq c^2 \tag{4.2}$$

for a constant $c \geq 0$.

Lemma 4.1. *Assume that f is a smooth function on (N, g) with $\nabla_i\nabla_j f = \lambda g_{ij}$ and that $(\widetilde{N}, \widetilde{g})$ is the associated warped product manifold. Then there exists a constant d, depending only on $n = \dim(N)$ such that*

$$|\widetilde{\nabla}\widetilde{R}|^2 \leq c^2 + d(\lambda^2 + |\nabla\lambda|^2 + \max\{-K_2, K_1\}^2)|\nabla f|^2. \tag{4.3}$$

Furthermore the sectional curvatures $\widetilde{\sigma}$ of $\widetilde{N}$ are bounded between

$$\min\{K_2, \lambda - |\nabla f|^2\} \leq \widetilde{\sigma} \leq \max\{K_1, \lambda\}. \tag{4.4}$$

Proof: From Proposition 2.3 we get

$$\begin{aligned} |\widetilde{\nabla}\widetilde{R}|^2 &= |\nabla R|^2 + 4e^{-4f}|\widetilde{\nabla}_0\widetilde{R}_{0ijk}|^2 + 4e^{-4f}|\widetilde{\nabla}_k\widetilde{R}_{0i0j}|^2 \\ &= |\nabla R|^2 + 16n\lambda^2|\nabla f|^2 - 16\lambda\mathrm{Ric}(\nabla f, \nabla f) + 4|\nabla^n f R_{nijk}|^2 \\ &+ 4n|\nabla\lambda|^2 - 16\lambda\langle\nabla\lambda, \nabla f\rangle. \end{aligned}$$

We apply Schwarz' inequality to the last terms and are done with (4.3). Inequality (4.4) is a direct consequence of (2.8) and (2.10). □

Corollary 4.2. *Let M be a submanifold contained in a simply connected manifold N and assume that $\Omega \subset M$ is a given subset. If $X = \nabla f$, then on Ω*

$$|\widetilde{\nabla}\widetilde{R}|^2 \leq c^2 + d(\sup_\Omega(\lambda^2 + |\nabla\lambda|^2) + \max\{-K_2, K_1\}^2)\mathrm{sup}_\Omega|X|^2 =: \widetilde{c}^2(\Omega)$$

$$\widetilde{K}_2(\Omega) := \min\{K_2, \inf_\Omega \lambda - \mathrm{sup}_\Omega|X|^2\} \leq \widetilde{\sigma} \leq \max\{K_1, \sup_\Omega \lambda\} =: \widetilde{K}_1.$$

Let us define

$$\gamma(\Omega, p) := (\widetilde{c}^{\frac{2}{3}}(\Omega) + \widetilde{K}_1 - \widetilde{K}_2(\Omega) + \max\{-\widetilde{K}_2(\Omega), 0\})^{\frac{p}{2}} \leq \infty. \tag{4.5}$$

Lemma 4.3. *If $u \in C_0^\infty(\Omega)$ for $\Omega \subset M$, then we have for any $h \in C^\infty(M)$*

$$\int_M hu^p d\mu = \int_{[0,1]\times M} h\widetilde{u}^p d\widetilde{\mu}, \tag{4.6}$$

with $\widetilde{u} := ue^{-\frac{f}{p}}$.
Proof: This follows directly from $d\widetilde{\mu} = e^f ds d\mu$. □

Theorem 4.4. *Assume that Ω is a domain in a stable conformal soliton M and that $\gamma(\Omega, p) < \infty$. Further assume that N is simply connected and that M is a hypersurface of dimension m. Then for each $p \in [4, 4 + \sqrt{\frac{8}{m+1}})$ and for each non-negative smooth function u with compact support in Ω the following inequality holds*

$$\int_M |A|^p u^p \leq \beta \int_M [|Du|^p + (\gamma(\Omega, p) + |X|^p)u^p] d\mu, \tag{4.7}$$

where β is a constant depending only on m and p and $\gamma(\Omega, p)$ is defined as in (4.5).
Proof: Since N is simply connected we can find a function f with $\nabla f = X$ and $\nabla_i \nabla_j f = \lambda g_{ij}$. First note that Theorem 3.1 and Lemma 3.5 imply that $\widetilde{\Omega} := \mathrm{R} \times \Omega$ is a stable minimal domain in $\widetilde{N}$. Lemma 4.3 gives

$$\int_M |A|^p u^p d\mu = \int_{[0,1]\times M} |A|^p \widetilde{u}^p d\widetilde{\mu} \leq \int_{[0,1]\times M} |\widetilde{A}|^p \widetilde{u}^p d\widetilde{\mu}, \tag{4.8}$$

with $\widetilde{u} := ue^{-\frac{f}{p}} \in C^\infty_{0,\mathrm{sym}}([0,1]\times\Omega)$. Since the gradient of a function $\widetilde{u} \in C^\infty_{0,\mathrm{sym}}([0,1]\times \Omega)$ is tangent to Ω, i.e. contains no components in the fiber directions $[x]$, we see that partial integration over $[0,1] \times M$ still yields no boundary terms. Then Theorem 1 in [15] and Corollary 4.2 implies

$$\int_{[0,1]\times M} |\widetilde{A}|^p \widetilde{u}^p d\widetilde{\mu} \leq \widetilde{\beta} \int_{[0,1]\times M} (|\widetilde{D}\widetilde{u}|^p + \gamma(\Omega, p)\widetilde{u}^p) d\widetilde{\mu} \tag{4.9}$$

for any function $\widetilde{u} \in C^\infty_{0,\mathrm{sym}}([0,1]\times\Omega)$, where $\widetilde{\beta}$ depends only on m and p. We compute

$$\begin{aligned} |\widetilde{D}\widetilde{u}|^2 &= |D(ue^{-\frac{f}{p}})|^2 \\ &\leq 2(|e^{-\frac{f}{p}} Du|^2 + |uDe^{-\frac{f}{p}}|^2) \\ &= 2e^{-\frac{2f}{p}}(|Du|^2 + \frac{u^2}{p^2}|Df|^2). \end{aligned}$$

And then

$$|\widetilde{D}\widetilde{u}|^p \leq d_p e^{-f}(|Du|^p + u^p |Df|^p) \tag{4.10}$$

with a constant d_p depending only on p. Since $|Df|^2 \le |\nabla f|^2 = |X|^2$ we arrive at

$$|\widetilde{D}\widetilde{u}|^p \le d_p e^{-f}|Du|^p + d_p \widetilde{u}^p |X|^p. \tag{4.11}$$

Summarizing (4.8), (4.9) and (4.11) we see that

$$\begin{aligned}\int_M |A|^p u^p d\mu &\le \widetilde{\beta}\int_{[0,1]\times M} d_p e^{-f}|Du|^p + (\gamma(\Omega,p) + d_p|X|^p)\widetilde{u}^p d\widetilde{\mu} \\ &= \widetilde{\beta}\int_M d_p|Du|^p + (\gamma(\Omega,p) + d_p|X|^p)u^p d\mu \\ &\le \beta\int_M |Du|^p + (\gamma(\Omega,p) + |X|^p)u^p d\mu, \qquad (4.12)\end{aligned}$$

where $\beta := \widetilde{\beta} \cdot \max\{d_p, 1\}$ depends only on m and p. □

Remark 4.5. Note that for a stable translating soliton in R^{m+1} we have $\widetilde{c} = 0 = \widetilde{K}_1 = \lambda$ and that the length of X is constant such that $\widetilde{K}_2 = -|X|^2$ and $\gamma(M,p) = (2|X|^2)^{\frac{p}{2}}$. Then inequality (4.7) reduces to

$$\int_M |A|^p u^p \le \beta_2 \int_M [|Du|^p + u^p] d\mu, \tag{4.13}$$

where β_2 is a different constant depending only on m and p.

Let's assume that there exists a constant R_0 with $0 < R_0 \le \infty$ and a family of subsets $\{B_R\}_{R\in(0,R_0)}$ defined by

$$B_R := \{x \in M : r(x) \le R\}, \tag{4.14}$$

where r is an arbitrary distance function, i.e. a Lipschitz function on M with $|Dr|^2 \le 1$, a.e. on M. In addition assume that each B_R is compact and that

$$M = \bigcup_{R\in(0,R_0)} B_R. \tag{4.15}$$

For the remainder of this article we will assume that N is simply connected, so that the warped product metric is well defined on $\widetilde{N} = \mathrm{R} \times N$. Any distance function r on M can be extended to a distance function $\widetilde{r}$ on $\widetilde{M} = \mathrm{R} \times M$ in an obvious way, namely by setting

$$\widetilde{r}(s,x) := r(x), \ \forall (s,x) \in \widetilde{M}. \tag{4.16}$$

We call $\widetilde{r}$ the extension of r.

Corollary 4.6. *In addition to the assumptions made in Theorem 4.4 we assume here that $\gamma(M,p) < \infty$. Then*

$$\int_{B_{\theta R}} |A|^p d\mu \le \widehat{\beta} R^{-p} |B_R| \tag{4.17}$$

for all $R \in (0, R_0), \theta \in (0,1), p \in (0, 4 + \sqrt{\frac{8}{m+1}})$*, where* $\widehat{\beta} = \widetilde{\beta}[(1-\theta)^{-p} + R^p \gamma(M,p)]$ *with the constant* $\widetilde{\beta}$ *as in the proof for Theorem 4.4 and* $|B_R|$ *is the m-dimensional volume.*

Proof: Extend r to $\widetilde{r}$, use $|A|^2 \leq |\widetilde{A}|^2$ and apply (4.9) and inequality (2.9) in [15]. □

Corollary 4.7. *Suppose* N *is simply connected, complete and that* $\max\{K_1, \lambda\} \leq 0$*. Then if* $m \leq 4$ *and*

$$R^2(\widetilde{c}^{\frac{2}{3}} + |\widetilde{K}_2|) + R^{-(m+1)}|B_R| \leq \beta_0 \tag{4.18}$$

we have

$$\sup_{B_{\theta R}} |A|^2 \leq \beta_3 R^{-2} \tag{4.19}$$

for all $\theta \in (0,1)$*, where* β_3 *is a constant that depends only on* β_0, θ *and* m*.*

Proof: Under these assumptions we conclude that $[0,1] \times N$ equipped with the warped product metric satisfies all assumptions made in Theorem 3 of [15] except for the completeness condition which is not true in the fiber directions. But since we are only interested in the symmetric behavior we can weaken this condition so that $[0,1] \times N$ needs to be complete only in the vertical component N. Then we use $|A|^2 \leq |\widetilde{A}|^2$ and the extension $\widetilde{r}$ to proceed in the same way as in [15] to derive the desired result. □

Remark 4.8. There are a couple of open questions. First of all one needs to find new examples of stable solitons. In [11] it has been shown that the only self-similarly evolving compact and mean convex soliton in R^{n+1} must be the standard sphere. Is it true that beside the sphere any compact soliton in R^{n+1} is unstable? We do not know whether the grim reaper curve $y = -\log(\cos(x))$ is stable in the sense of Definition 3.6. Here one has $X = \frac{\partial}{\partial y}$, $\lambda = 0$ and

$$|A|^2 + \mathrm{Ric}(\nu,\nu) - \frac{1}{4}H^2 - \frac{m-2}{2}\lambda - \frac{1}{4}|X|^2 = \frac{3H^2-1}{4} \tag{4.20}$$

and the stability condition reads

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3(\cos(x))^2 - 1}{4\cos(x)} u(x)^2 dx \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)(u'(x))^2 dx \tag{4.21}$$

for all $u \in C_0^\infty((-\frac{\pi}{2}, \frac{\pi}{2}))$.

If $\widetilde{M_t} \subset \widetilde{N}$ is a smooth family of immersions that are symmetric in the fiber direction and that evolve according to the MCF in $\widetilde{N}$, then the cross sections $M_t := \pi(\widetilde{M_t}) \subset N$ (π being the projection) evolve by

$$\frac{d}{dt}F_t = \overrightarrow{H} - X^\perp. \tag{4.22}$$

This follows from the calculations in [16]. If ϕ_s is the 1-parameter family of conformal deformations belonging to X, then the rescaled immersions $\widehat{F}_t := \phi_t F_t$ solve the MCF equation in N with a different time scale and an additional tangential deformation (that does not effect the geometry and merely corresponds to a diffeomorphism on the evolving hypersurface). Thus we see that the mean curvature flow of the lift $\widetilde{M} \in \widetilde{N}$ is just the lift of a rescaled MCF for M in N.

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