Note on the spectrum of the Hodge-Laplacian for k-forms on minimal Legendre submanifolds in S^{2n+1}

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Abstract

Given a minimal Legendre immersion L in S^{2n+1} and $n \geq k \geq 1$ we prove that $n+1-k$ is an eigenvalue of the Hodge-Laplacian acting on k and $(k-1)$-forms on L. In particular we show that the eigenspaces $\mathrm{Eig}_k(n+1-k)$ and $\mathrm{Eig}_{k-1}(n+1-k)$ are at least of dimension $\binom{n}{k}$

1 Preliminaries

A contact manifold (of restricted type)[1] (M, λ) is an odd-dimensional manifold of dimension $2n+1$ together with a one-form λ such that $\lambda \wedge (d\lambda)^n$ defines a volume form on M. One observes that a contact manifold is orientable and that the contact form λ defines a natural orientation.

Assume now that (M, λ) is a given contact manifold of dimension $2n+1$. λ defines a $2n$-dimensional vector bundle ξ over M, where at each point $p \in M$ the fiber ξ_p of ξ is given by

$$\xi_p = \ker \lambda_p.$$

Moreover, since $\lambda \wedge (d\lambda)^n$ is a volume form, we see that

$$\omega := d\lambda_{|\xi \oplus \xi}$$

is a closed non degenerate two-form and hence defines a symplectic product on ξ so that (ξ, ω) becomes a symplectic vector bundle. Since the dimension of M is odd, the two-form $d\lambda$ must be degenerate on TM. Therefore one obtains a line bundle l over M via the definition

$$l_p := \{V \in T_pM | \; d\lambda(V, W) = 0 \; \forall \; W \in \xi_p\}$$

[1] More generally a contact manifold M is a differentiable manifold of odd dimension $2n+1$ with a completely nonintegrable distribution ξ of hyperplanes in the tangent space. Locally such hyperplane fields can be described as the kernel of a nonvanishing one-form λ. The nonintegrability then implies that $\lambda \wedge (d\lambda)^n$ locally defines a volume form. If this one-form λ exists globally then we speak of a contact manifold of restricted type.

The Reeb vector field (sometimes called characteristic vector field) X_λ is given by the natural section X_λ in l defined by

$$\lambda(X_\lambda) = 1 \ , \ X_\lambda \lrcorner d\lambda = 0.$$

Thus a contact form λ on an odd-dimensional manifold M of dimension $2n+1$ defines a splitting of the tangent bundle TM into a line bundle l with canonical section X_λ and a symplectic vector bundle (ξ, ω):

$$TM = (l, X_\lambda) \oplus (\xi, \omega).$$

We denote the projection of TM along l by π, i.e.

$$\begin{aligned} \pi \quad &: \quad TM \to \xi, \\ \pi(V) \quad &:= \quad V - \lambda(V) X_\lambda. \end{aligned}$$

A submanifold L of a $(2n+1)$-dimensional contact manifold (M, λ) is called isotropic if it is tangent to ξ, i.e. if $\lambda_{|L} = 0$. This implies that $d\lambda_{|L} = \omega_{|L} = 0$ also. An isotropic submanifold L of maximal dimension n is called Legendrian. The standard example of a contact manifold is the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$, where for the outward pointing unit normal ν one defines $X_\lambda := J(\nu)$, J denoting the standard complex structure on $\mathbb{R}^{2n+2}$. The contact form λ is then given by the dual one-form of X_λ. One easily observes that a Legendrian submanifold in S^{2n+1} is minimal if and only if the corresponding cone in $\mathbb{R}^{2n+2}$ is a minimal Lagrangian cone. Minimal Lagrangian submanifolds are important in elasticity and in string theory (see [4] and [6]). Since minimal Lagrangian cones are relevant in the study of singularities for minimal Lagrangian submanifolds it is clear that minimal Legendrian submanifolds of S^{2n+1} are important as well. In [5] we proved that the restriction of a parallel, anti-compatible k-form on a Kähler manifold to a minimal Lagrangian immersion becomes a harmonic k-form. In particular this result implied the nonexistence of minimal, orientable, closed Lagrangian immersions L in hyperKähler manifolds if the second Betti number of L equals 0. In this paper we will follow this idea to show that any parallel k-form on $\mathbb{R}^{2n+2}$ that satisfies an anti-compatibility condition restricts to a eigenform of the Hodge-Laplacian with eigenvalue given by $(n+1-k)$. We note that for a general Riemannian manifold the spectral theory for the Hodge-Laplacian is not well developed. For a generic Riemannian manifold it is usually impossible to compute the spectrum. Nevertheless a detailed description of the spectrum is often useful in the study of special properties for the underlying metric like curvature estimates, closed geodesics etc. The spectrum and the multiplicities of eigenvalues depend on the metric. Therefore it seems to be rather mysterious that minimal Legendre immersions in S^{2n+1} all posses at least some identical eigenvalues. Perhaps they are even isospectral in some cases.

1.1 The geometry of Legendrian submanifolds in S^{2n+1}

Let L be a Legendrian submanifold of S^{2n+1} equipped with the standard metric $\bar{g}$ and with Levi-Civita connection $\bar{\nabla}$. The induced metric and connection on L will be denoted by $g = \langle \ , \ \rangle, \nabla$ resp., i.e. $\nabla_X Y := (\bar{\nabla}_X \bar{Y})^T$ for any $X, Y \in TL$ and $\bar{Y}$ being any extension of Y in S^{2n+1}. Here ^{T} denotes the orthogonal projection onto the tangent bundle TL.

Recall that the second fundamental tensor A of L in S^{2n+1} is defined by the symmetric map

$$\begin{aligned} A &: TL \otimes TL \to NL, \\ A(X,Y) &:= (\bar{\nabla}_X \bar{Y})^{\perp}, \end{aligned}$$

where $^{\perp}$ denotes the orthogonal projection onto the normal bundle NL. Hence the Gauss formula

$$\bar{\nabla}_X \bar{Y} = \nabla_X Y + A(X,Y)$$

holds. The second fundamental form h_ν with respect to a normal vector ν at $p \in L$ is then given by the symmetric map

$$\begin{aligned} h_\nu &: T_pL \times T_pL \to \mathbb{R}, \\ h_\nu(X,Y) &:= -\langle \nu, A(X,Y) \rangle. \end{aligned}$$

Since L is Legendrian we observe that J provides an isomorphism between tangent vectors $V \in TL$ and those normal vectors ν being in ξ. Hence the normal bundle NL can be split into the direct sum

$$NL = l \oplus J(TL)$$

via the map

$$\begin{aligned} \phi_p &: \mathbb{R} \times T_pL \to N_pL, \\ \phi_p(f,V) &:= fX_\lambda(p) + JV. \end{aligned}$$

Then the second fundamental forms can be described by the set of tensors

$$\begin{aligned} h(Z,X,Y) &:= -\langle JZ, A(X,Y) \rangle, \\ k(X,Y) &:= -\langle X_\lambda, A(X,Y) \rangle, \end{aligned}$$

where $X, Y, Z \in TL$. As usual these tensors are symmetric in X and Y. As X_λ is globally defined on S^{2n+1} one can view k as the restriction of the exterior tensor

$$\bar{k}(X,Y) := \langle \bar{\nabla}_X X_\lambda, Y \rangle$$

to L. Since $\langle X_\lambda, Y \rangle = \lambda(Y),\ \forall\ Y \in TS^{2n+1}$ we see that

$$\bar{k}(X,Y) = \bar{\nabla}_X \lambda(Y).$$

We also recall the following: Let $Y, X_1, \ldots, X_k$ be in $TL \subset TS^{2n+1}$ and assume that T is a tensor of degree k on S^{2n+1}. Then

$$\begin{aligned} \bar{\nabla}_Y T(X_1,\ldots,X_k) &= \nabla_Y T_{|L}(X_1,\ldots,X_k) \qquad (1) \\ &- \sum_{i=1}^{k} T(X_1,\ldots,X_{i-1},A(Y,X_i),X_{i+1},\ldots,X_k). \end{aligned}$$

We apply this formula to $d\lambda$. Since $d\lambda_{|L} = \omega_{|L} = 0$ we get

$$\bar{\nabla}_X d\lambda(Y,Z) = -d\lambda(A(X,Y),Z) - d\lambda(Y,A(X,Z)).$$

On the other hand

$$d\lambda(V,W) = \langle JV, W \rangle,\ \forall\ V,W \in TS^{2n+1}$$

gives

$$\bar{\nabla}_X d\lambda(Y,Z) = h(Y,X,Z) - h(Z,X,Y),\ \forall\ X,Y,Z \in TL.$$

and since by the Legendrian condition we must have $(\bar{\nabla}_X d\lambda)_{|L} = 0$, we see that h is fully symmetric. Now let K be a parallel k-form on $\mathbb{R}^{2n+2}$, $k \geq 1$ and define $\kappa(V_1, \ldots, V_{k-1}) := K(\nu, V_1, \ldots, V_{k-1})$ as a $(k-1)$-form on S^{2n+1}. We can also restrict K to S^{2n+1} (also denoted by K). Since the second fundamental form of S^{2n+1} equals its first fundamental form and K is parallel on $\mathbb{R}^{2n+2}$ we obtain from equation (1) that

$$\bar{\nabla}_i K_{j_1 \ldots j_k} = \sum_{l=1}^{k} (-1)^l g_{ij_l} \kappa_{j_1 \ldots j_{l-1} j_{l+1} \cdots j_k} \tag{2}$$

and also

$$\bar{\nabla}_i \kappa_{j_1 \ldots j_{k-1}} = K_{ij_1 \ldots j_{k-1}}. \tag{3}$$

In this paper we will use the convention

$$dx^{j_1} \wedge \cdots \wedge dx^{j_k} := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) dx^{j_{\sigma(1)}} \otimes \cdots \otimes dx^{j_{\sigma(k)}}.$$

With this convention we have

$$K = K_{j_1 \ldots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k} = K_{j_1 \ldots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

and

$$d\kappa = \sum_{i=0}^{k-1} (-1)^{(i+1)(k-1)} \kappa_{j_{\tau^i(1)} \cdots j_{\tau^i(k-1)}, j_{\tau^i(k)}} dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

where $\tau \in S_k$ is the right shift given by $\tau(l) = l - 1$ modulo k. Then equation (3) just means that

$$d\kappa = K. \tag{4}$$

Next we want to compute the Hodge Laplacian of $\kappa_{|L}$ under the assumption that L is minimal Legendrian. First note that for a minimal Legendrian immersion in S^{2n+1} the second fundamental form k w.r.t. the normal vector X_λ must vanish identically. This follows from the following facts: First we can compute exactly as above that $\bar{\nabla}_i \lambda_j = \omega_{ij}$, where ω_{ij} is the restriction of the standard symplectic structure on $\mathbb{R}^{2n+2}$ to S^{2n+1}. Hence $\bar{\nabla}\lambda$ is anti-symmetric. On the other hand we have already seen that $k = \bar{\nabla}\lambda_{|L}$. And since the second fundamental form is symmetric we observe that $\bar{\nabla}\lambda_{|L} = k = 0$. Again using formula (1) we get

$$\nabla_i K_{j_1 \ldots j_k} = \bar{\nabla}_i K_{j_1 \ldots j_k} - \sum_{l=1}^{k} h^m{}_{ij_l} K_{j_1 \ldots j_{l-1} \underline{m} j_{l+1} \cdots j_k},$$

where an underlined index means that one has to apply J, e.g. $\kappa_{i\underline{j}}$ is shorthand for $\kappa(e_i, J(e_j))$. Consequently if we take into account (2), then we obtain

$$\nabla_i K_{j_1 \ldots j_k} = \sum_{l=1}^{k} (-1)^l g_{ij_l} \kappa_{j_1 \ldots j_{l-1} j_{l+1} \cdots j_k} - \sum_{l=1}^{k} h^m{}_{ij_l} K_{j_1 \ldots j_{l-1} \underline{m} j_{l+1} \cdots j_k} \tag{5}$$

In the same way we obtain

$$\nabla_i \kappa_{j_1 \ldots j_{k-1}} = K_{ij_1 \ldots j_{k-1}} - \sum_{l=1}^{k-1} h^m{}_{ij_l} \kappa_{j_1 \ldots j_{l-1} \underline{m} j_{l+1} \cdots j_{k-1}}. \tag{6}$$

Now we can prove

Theorem 1.1 *Assume that $k \geq 1$ and K is a parallel k-form on $\mathbb{R}^{2n+2}$. Further assume that K satisfies the anti-compatibility condition*

$$K(JV, W, Z_1, \ldots, Z_{k-2}) = -K(JW, V, Z_1, \ldots, Z_{k-2}), \ \forall V, W \in \mathbb{R}^{2n+2}$$

provided $k \geq 2$. Then the restrictions of K and $\kappa = \nu \lrcorner K$ to any minimal Legendrian immersion L of S^{2n+1} satisfy the equations

$$\begin{aligned} \Delta K_{|L} &= (n+1-k)K_{|L} \\ \Delta \kappa_{|L} &= (n+1-k)\kappa_{|L}. \end{aligned}$$

Hence if they do not vanish on L they are eigenforms of the Hodge-Laplacian with eigenvalues given by $n+1-k$.

Proof: Using equation (5) we obtain

$$\begin{aligned} (d^\dagger K)_{j_2 \ldots j_k} &= \nabla_i K_{ij_2 \ldots j_k} \\ &= (-n+k-1)\kappa_{j_2 \ldots j_k} \\ &- H^m K_{\underline{m} j_2 \ldots j_k} - \sum_{l=2}^{k} h^{mi}{}_{j_l} K_{ij_2 \ldots j_{l-1} \underline{m} j_{l+1} \ldots j_k}, \end{aligned}$$

where $H^m = h^{mi}{}_i$ is one of the mean curvatures of L. The last term appears only in the case $k \geq 2$. Since L is minimal we have $H^m = 0$. On the other hand the anti-compatibility condition implies that

$$h^{mi}{}_{j_l} K_{ij_2 \ldots j_{l-1} \underline{m} j_{l+1} \ldots j_k}$$

is the contraction of an anti-symmetric tensor with a symmetric tensor. Hence the last term vanishes also. In summary we obtain

$$d^\dagger K_{|L} = (-n+k-1)\kappa_{|L}. \tag{7}$$

With the same method we obtain

$$d^\dagger \kappa_{|L} = 0 \tag{8}$$

Moreover

$$d\kappa_{|L} = K_{|L} \tag{9}$$

since this is true on S^{2n+1} and the differential and inclusion operator commute. Since K is parallel on $\mathbb{R}^{2n+2}$ it must me closed. Again by the general principle of interchanging d and the inclusion operator we obtain

$$dK_{|L} = 0. \tag{10}$$

Now the Hodge-Laplacian is given by

$$\Delta = -(dd^\dagger + d^\dagger d)$$

from which we conclude the theorem. □

Theorem 1.2 *Let $Eig_k(c)$ denote the eigenspace w.r.t. the real number c for the Hodge-Laplacian acting on k-forms on L with $n \geq k \geq 1$. Then*

$$\begin{aligned} dim\big(Eig_{k-1}(n+1-k)\big) &\geq \binom{n}{k}, \\ dim\big(Eig_k(n+1-k)\big) &\geq \binom{n}{k}. \end{aligned}$$

Proof: Fix an arbitrary point $p \in L$ and let $e_1, \dots, e_n$ be an orthonormal basis of T_pL. Any of the $\binom{n}{k}$ k-forms $\eta := e_{i_1} \wedge \cdots \wedge e_{i_k}$ on T_pL extends to a parallel k-form on $\mathbb{R}^{2n+2}$. In the case $k \geq 2$ we define

$$\bar{\eta}(V, W, Z_1, \dots, Z_{k-2}) := \eta(V, W, Z_1, \dots, Z_{k-2}) - \eta(JV, JW, Z_1, \dots, Z_{k-2})$$

and obtain a parallel k-form on $\mathbb{R}^{2n+2}$ that satisfies the anti-compatibility condition. Since the Legendre condition implies that any tangent vector on L is mapped to a normal vector under J we see that $\bar{\eta}_{|T_pL} = \eta_{|T_pL} \neq 0$. Then we can use Theorem 1.1 to derive the second inequality. The first inequality follows from the second if we use the $(k-1)$-forms $\lambda := \nu \lrcorner \eta$ and take into account that $d\lambda = \eta$ means that $\eta \neq 0$ implies $\lambda \neq 0$. □

Remark: *To prove that $dim\big(Eig_0(n)\big) \geq n$ and $dim\big(Eig_1(n)\big) \geq n$ one does not need the Legendre condition at all. Any of the coordinate functions y^α on $\mathbb{R}^{2n+2}$ restricts to a function f on a minimal immersion $L^p, 1 \leq p \leq 2n+1$ in S^{2n+1} such that $\Delta f = pf$ and at least p of these coordinate functions must be nontrivial.*

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