Evolution of hypersurfaces in central force fields

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Abstract

We consider flows of hypersurfaces in $\mathbb{R}^{n+1}$ decreasing the energy induced by radially symmetric potentials. These flows are similar to the mean curvature flow but different phenomena occur. We show for a natural class of potentials that hypersurfaces converge smoothly to a uniquely determined sphere if they satisfy a strengthened starshapedness condition at the beginning.

1 Introduction

Consider a smooth family $F_t : M \to \mathbb{R}^{n+1} \setminus \{0\}$ of oriented hypersurfaces $M_t := F_t(M)$ that evolve according to

$$\frac{d}{dt}F = -(H - \phi(s)\langle F, \nu\rangle)\nu \equiv -f\nu, \tag{1}$$

where we have omitted the index t. Here, H denotes the mean curvature of M_t w. r. t. the outer unit normal ν (i. e. $H > 0$ for the standard sphere)

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and ϕ is a smooth, radially symmetric function (reflecting the presence of a central force) depending on $s := \frac{|F|^2}{2}$. More precisely ϕ is related to the potential v on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$v(s) := \exp\left(-\frac{n}{2}\int_1^s \frac{\eta(\sigma)}{\sigma} d\sigma\right), \quad \phi(s) = -\left.\frac{\frac{\partial}{\partial\sigma}v(\sigma)}{v(\sigma)}\right|_{\sigma=s} = \frac{n\eta(s)}{2s}$$

where $\eta : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function. The total potential energy $V(F)$ of the hypersurface $F(M)$ will then be defined as

$$V(F) := \int_M v(s(F)) d\mu.$$

A physical interpretation of our flow is as follows. Suppose we have an electrically charged membrane with a constant charge per area. If this surface is in a radially symmetric potential, then the energy of the system is given by the energy in the exterior potential plus other terms, for example are there forces between different parts of the surface, which we neglect. The negative gradient flow for this energy is given by

$$\frac{d}{dt}F = -v(H - \phi(s)\langle F, \nu\rangle)\nu,$$

so we see that the stationary solutions of (1) coincide with the stationary solutions of the negative gradient flow. We will prove the following main theorem

Theorem 1.1 *Let $\widetilde{F} : S^n \to \mathbb{R}^{n+1}$ be a smooth embedding of a strictly star-shaped hypersurface such that*

$$0 < s_- \leq s = \frac{|\widetilde{F}|^2}{2} \leq s_+ < \infty.$$

Further assume that η satisfies

$$\exists\, s_0 \in [s_-, s_+] \text{ with } \eta(s_0) = 1, \tag{2}$$

$$\eta'(s) < -\frac{2}{n}c_\eta < 0 \quad \forall\, s \in [s_-, s_+] \tag{3}$$

for a constant c_η *and that the strengthened starshapedness condition*

$$\frac{f}{\langle \widetilde{F}, \nu \rangle} < c_\eta \tag{4}$$

holds for $\widetilde{F}$*. Then (1) admits an embedded solution* F_t *for all* $t \geq 0$ *with* $F_0 = \widetilde{F}$ *and* $S_t := F_t(S^n)$ *converges exponentially in the* C^∞*-topology to a stable sphere centered at the origin with radius* $r_0 = \sqrt{2s_0}$*.*

Remark: Condition (4) can be easily satisfied.

Let $\alpha > 0$, $\beta > 0$ and assume that ϕ has the form $\phi(s) = \frac{\alpha}{s^{1+\beta}}$. Let $\tilde{F}$ be as in the main theorem, $r_0 = \max \left|\tilde{F}\right|$. At the point where this maximum is attained we have with $s_0 = \frac{1}{2} r_0^2$

$$c_\eta > \frac{f}{\left\langle \tilde{F}, \nu \right\rangle} \geq \frac{1}{s_0} \left(\frac{n}{2} - \frac{\alpha}{s_0^\beta} \right).$$

On the other hand $\eta'(s_0) < -\frac{2}{n} c_\eta$ implies

$$\frac{\alpha\beta}{s_0^{1+\beta}} > c_\eta$$

so we obtain

$$s_0 < \left(\frac{2}{n} \alpha (1 + \beta) \right)^{\frac{1}{\beta}}$$

and see that $r_0 \leq R(n, \alpha, \beta)$.

We have chosen the special ansatz for the potential v as the stability of a stationary sphere of radius $r_0 = \sqrt{2s_0}$ is equivalent to the simple condition $\eta'(s_0) < 0$.

We wish to explain the condition (4) assumed for the embedding $\tilde{F}$. First we show, that the flow does not preserve convexity nor starshapedness during the evolution (so a somehow different initial condition is needed for the smooth long-time existence), then we give a geometric interpretation of (4).

We assume a potential v that tends to infinity for $s \to 0$ and decays for $s \to \infty$. Then a convex hypersurface does not need to stay convex during its evolution. We take a sphere of large radius but assume that the origin is very close to the surface. At the beginning of the evolution the surface is repelled apart from the origin where it is very close to the origin but moves relatively slowly at a large distance from the origin. As a sphere of large radius is nearly flat near a fixed point this destroys convexity.

A more elaborated example will show that starshapedness with respect to the origin is not preserved, too. We describe our surface in polar coordinates as follows. The height above the unit sphere is given by a positive constant times the characteristic function of a small geodesic ball around a fixed point. The example is obtained if we smooth out this situation slightly. Figure 1 shows a cross-section of this hypersurface.

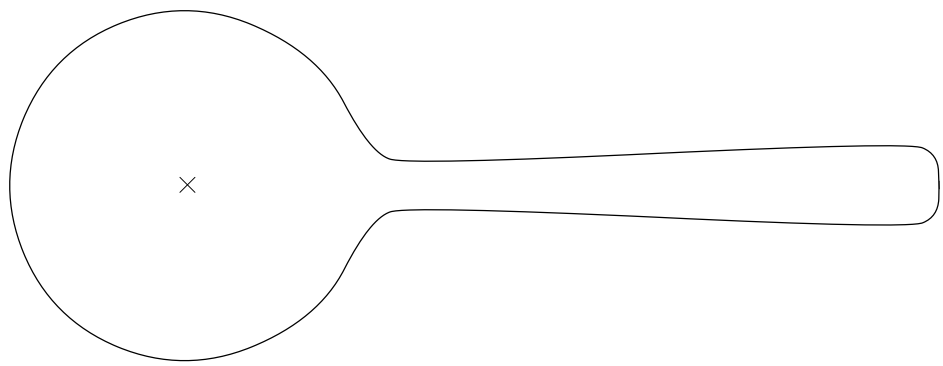

Figure 1: Starshaped example

Rotating it around the horizontal axis gives the whole hypersurface. If the neck is thin, $n-1$ principal curvatures become very large there. The remaining negative principal curvature there is small compared to the others when the neck becomes small. Of course the non-spherical part has to be made longer simultaneously so that the curvature near the tip remains bounded. This construction yields for a small potential compared to the mean curvature that the motion of the surface is especially large at the neck where we assume that it shrinks so fast that not only the starshapedness is lost but also the evolution will become singular in finite time.

If we represent the hypersurface as graph $u|_{S^n}$ (this is possible for starshaped hypersurfaces with respect to the origin), then the time derivative of $\log u$ equals $-\frac{f}{\langle F,\nu\rangle}$. The change of coordinates $u \mapsto \log u$, however, is natural as the induced metric of the hypersurface in this new coordinates is very similar to the induced metric for graphs in Euclidean space, in fact, this shows the conformal equivalence of $S^n \times \mathbb{R}$ and $\mathbb{R}^{n+1} \setminus \{0\}$. More details can be found in the appendix.

Another interesting feature of our flow can be obtained for $\phi = 1$, a case which is not considered in our main theorem. Then the stationary solutions are characterized by $H = \langle F, \nu \rangle$ and these hypersurfaces shrink homothetically under the evolution of the mean curvature flow $\frac{d}{dt}F = -H\nu$.

We consider a potential with appropriate asymptotics as for example the potential induced by $\eta(s) = \frac{2}{n}\frac{1}{s}$. Then there are two possibilities to reduce the energy; the surface may move apart from the source of the potential or it may contract and thus reduce its area and its energy as the charge is assumed to be proportional to the area. When the surface encloses the origin, i. e. the source of the potential, a charged point, these two effects are opposite to each other and so it is reasonable to conjecture that the surface tends to a sphere for which both forces compensate each other when the topology allows this.

We wish to mention some further papers on related problems. In [1], [4], [6] and [7] the evolution of starshaped hypersurfaces for various curvature driven flow equations has been considered. In contrast to outward directed flows, where starshapedness usually is preserved, this fails for inward directed flows like the mean curvature flow. For these flows convexity is naturally preserved [2]. But even for inward directed flows we have nice properties for starshaped hypersurfaces [6].

The paper is organized as follows: In section 2 we introduce notations from differential geometry and compute the Euler-Lagrange equations, in section 3 we show how spheres can be used as barriers for our flow. In section 4 we derive evolution equations for geometric quantities, deduce a priori estimates and prove the smooth convergence to a stable sphere. Finally, we state additional properties of our flow in the appendix.

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2 Euler-Lagrange equations

We are interested in the first and second variation of V. To this end assume that $F_t : (-\epsilon, \epsilon) \times M \to \mathbb{R}^{n+1}$ is a smooth family of immersions of orientable hypersurfaces such that

$$\frac{d}{dt} F_t = -f\nu,$$

with a smooth function f depending on $t \in (-\epsilon, \epsilon)$ and ν denoting the outward unit normal. We also recall the Gauß formula, the equations of Gauß, Weingarten, Codazzi and Simons

Proposition 2.1

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \tag{5}$$

$$\nabla_i \nabla_j F = -h_{ij}\nu, \tag{6}$$

$$\nabla_i \nu = h_i{}^l \nabla_l F, \tag{7}$$

$$\nabla_k h_{ij} = \nabla_j h_{ik}, \tag{8}$$

$$\nabla_i \nabla_j H = \Delta h_{ij} - H h_i{}^l h_{lj} + |A|^2 h_{ij}, \tag{9}$$

$$2h^{ij} \nabla_i \nabla_j H = \Delta |A|^2 - 2|\nabla A|^2 - 2Z. \tag{10}$$

Here h_{ij} is the second fundamental form and

$$\begin{aligned} H &= g^{ij} h_{ij}, \\ |A|^2 &= g^{ij} g^{kl} h_{ik} h_{jl}, \\ C &= g^{ij} g^{kl} g^{st} h_{ik} h_{js} h_{lt}, \\ Z &= HC - |A|^4. \end{aligned}$$

g^{ij} is the inverse of the induced metric

$$g_{ij} := \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle.$$

Doubled indices are always summed from 1 to n, indices are raised and lowered with respect to the induced metric and ∇ denotes the Levi-Civita connection on M w. r. t. g_{ij}. In the sequel we won't distinguish between vectors $V \in T_x M$ and $DF(V) \in T_{F(x)}\mathbb{R}^{n+1}$ and we also use $\langle \cdot, \cdot \rangle$ both for the scalar product on $\mathbb{R}^{n+1}$ and on M. The calculations in [5] give

Proposition 2.2

$$\frac{d}{dt}g_{ij} = -2fh_{ij}\ , \tag{11}$$

$$\frac{d}{dt}d\mu = -fHd\mu\ , \tag{12}$$

$$\frac{d}{dt}\nu = DF(\nabla f) = g^{ij}\nabla_i f \nabla_j F\ , \tag{13}$$

$$\frac{d}{dt}h_{ij} = \nabla_i\nabla_j f - f h_i^{\ k} h_{kj}\ , \tag{14}$$

$$\frac{d}{dt}H = \Delta f + f|A|^2\ , \tag{15}$$

$$\frac{d}{dt}|A|^2 = 2h^{ij}\nabla_i\nabla_j f + 2fC. \tag{16}$$

In addition we get

$$\frac{d}{dt}v = -\frac{n\eta}{2s}v\frac{d}{dt}s = f\phi\langle F,\nu\rangle v. \tag{17}$$

Lemma 2.3 *The first and second variation for V and compactly supported $f \in C_0^\infty(M)$ are given by*

$$\frac{d}{dt}V = -\int_M fv(H - \phi\langle F,\nu\rangle)d\mu,$$

$$\begin{aligned}\frac{d^2}{dt^2}V &= -\int_M (H - \phi\langle F,\nu\rangle)\frac{d}{dt}(fvd\mu)\\ &+ \int_M v\big(|\nabla f|^2 - f^2(|A|^2 + \phi + \phi'\langle F,\nu\rangle^2)\big)d\mu.\end{aligned}$$

Proof: The first variation formula is a direct consequence of (12) and (17). For the second variation we use (13), (15) and compute

$$\begin{aligned}-vf\frac{d}{dt}(H - \phi\langle F,\nu\rangle) &= -vf\big(\Delta f + f|A|^2 + f\phi'\langle F,\nu\rangle^2 + f\phi - \phi\langle\nabla s,\nabla f\rangle\big)\\ &= -\mathrm{div}(vf\nabla f) + v\big(|\nabla f|^2 - f^2(|A|^2 + \phi + \phi'\langle F,\nu\rangle^2)\big)\end{aligned}$$

and the second variation formula follows from partial integration. □

Corollary 2.4 *The Euler-Lagrange equation for stationary hypersurfaces w. r. t. V is given by*

$$H - \phi\langle F, \nu\rangle = 0.$$

A stationary hypersurface is stable (resp. strictly stable) if and only if

$$\int_M \big(|\nabla f|^2 - f^2(|A|^2 + \phi + \phi'\langle F, \nu\rangle^2)\big) v d\mu \geq 0 \quad (resp. \ > 0)$$

for all smooth, compactly supported $f \not\equiv 0$.

Corollary 2.5 *Assume that s_0 satisfies $\eta(s_0) = 1$ and that $\eta'(s_0) < 0$. Then the sphere with center at the origin and radius $r_0 = \sqrt{2s_0}$ is strictly stable.*

Proof: Left to the reader. ◻

The negative gradient flow for V is given by $\frac{d}{dt}F = -v(H - \phi\langle F, \nu\rangle)\nu$. Since this is stationary if and only if $H - \phi\langle F, \nu\rangle = 0$ we can as well consider the modified gradient flow

$$\frac{d}{dt}F = -f\nu \tag{18}$$

with

$$f := H - \phi\langle F, \nu\rangle.$$

We will set $M_t := F_t(M)$. Then the results in [3] imply

Proposition 2.6 *(18) is a system of quasilinear parabolic equations and there exists a maximal time $0 < T \leq \infty$ such that (18) admits a smooth solution on $[0, T)$.*

3 Inclusion principle

Lemma 3.1 *Assume that the initial value F_0 of a smooth solution F_t, $0 \leq t < T$, of the flow equation (18) is embedded. Then F_t, $0 < t < T$, is also embedded.*

Proof: Let t_0 be the first time when the immersion fails to be an embedding, $F_{t_0}(x_0) = F_{t_0}(y_0)$, $x_0 \neq y_0$. We choose a local coordinate system such that $F_{t_0}(x_0) = F_{t_0}(y_0) = 0$ and the unit normals at the origin are $\pm e_{n+1}$. So we may write F_{t_0} around x_0 and y_0 as graph (u^1) and graph (u^2), respectively. The representation as a graph is possible for a small time interval $[t_0-\varepsilon, t_0+\varepsilon]$, $\varepsilon > 0$, too. We may assume that $u^1 > u^2$ for $t < t_0$. If we replace ν by $-\nu$ in the flow equation, H, $\langle F, \nu\rangle$ and ν change sign, so we may assume that the normals at (x_0, t_0) and (y_0, t_0) both equal e_{n+1}. We rewrite the geometric evolution equation as a parabolic evolution equation for u^1 and u^2 and obtain

$$\frac{\partial}{\partial t}(u^1 - u^2) = a^{ij}(u^1_{ij} - u^2_{ij}) + b^i(u^1_i - u^2_i) + c(u^1 - u^2). \tag{19}$$

We remark that this equation is parabolic as H is elliptic for every smooth solution, i. e. especially for any function $\tau u^1 + (1-\tau)u^2$, $0 \leq \tau \leq 1$. From the strong maximum principle we deduce, that u^1 and u^2 coincide locally and furthermore, that these two functions have locally to be equal before $t = t_0$. This contradicts our assumption about t_0. □

Corollary 3.2 *Let F be a smooth immersed solution of (18) and $\tilde{F}$ be an immersed solution of this evolution equation. If $\tilde{F}$ is contained in a connected component of $\mathbb{R}^{n+1} \setminus F$ or in the closure of such a component at the beginning of the evolution, then this remains true during the evolution.*

Proof: As the geometric situation at the first point of contact is similar to the situation in the proof of Lemma 3.1, we can argue as we have done there. If F and $\tilde{F}$ touch for $t = 0$ then the strong maximum principle yields that they are disjoint at least for small positive times unless there are connected components of F and $\tilde{F}$ which coincide for $t = 0$. □

The following barrier argument will become important in the sequel.

Lemma 3.3 *Assume s_-, s_0, s_+ are positive numbers such that $s_- < s_0 < s_+$ and that η satisfies*

$$\begin{aligned} \eta(s_0) &= 1, \\ \eta'(s) &< 0 \quad \forall\, s \in [s_-, s_+]. \end{aligned}$$

Then under our flow all centered spheres of radius $r = \sqrt{2s}$ *with* $s \in [s_-, s_+]$ *converge to the stable sphere with radius* $r_0 = \sqrt{2s_0}$*. Moreover, if* $M_t, t \geq 0$*, is a smooth evolution of hypersurfaces such that* M_0 *is contained in*

$$\{y \in \mathbb{R}^{n+1} | r_- := \sqrt{2s_-} < |y| < r_+ := \sqrt{2s_+}\},$$

then M_t *will be contained in*

$$\{y \in \mathbb{R}^{n+1} | r_-(t) < |y| < r_+(t)\},$$

where $r_-(t)$ *and* $r_+(t)$ *are the radii for the evolutions of the inner and outer sphere. If* M_t *exist for all* $t \geq 0$*, then* M_t *converges pointwise to the stable sphere of radius* r_0*.*

Proof: The first part is an easy consequence of

$$\frac{d}{dt} s = n(\eta - 1) \tag{20}$$

which holds for centered spheres, see the independently proven Lemma 4.2. The remainder then follows from the inclusion principle 3.2. □

Remark: From (20), $\eta(s_0) = 1$ and $\eta'(s_0) < 0$ we deduce that the convergence in Lemma 3.3 is even exponentially.

4 Convergence to a stable sphere

Lemma 4.1 *The evolution equation for* f *is*

$$\frac{d}{dt} f = \Delta f - \phi \langle \nabla s, \nabla f \rangle + f(|A|^2 + \phi + \phi' \langle F, \nu \rangle^2).$$

Proof: This has been calculated in the proof of Lemma 2.3. □

Lemma 4.2 s *satisfies*

$$\frac{d}{dt} s = \Delta s - \phi |\nabla s|^2 + 2s\phi - n,$$

Proof: First we compute

$$\nabla_i s = \langle F, F_i \rangle$$

with $F_i := \frac{\partial F}{\partial x^i}$. Then the Gauß formula implies

$$\nabla_i \nabla_j s = g_{ij} - \langle F, \nu \rangle h_{ij}.$$

Therefore

$$\Delta s = n - H\langle F, \nu \rangle = n - f\langle F, \nu \rangle - \phi \langle F, \nu \rangle^2.$$

On the other hand

$$\frac{d}{dt} s = \langle F, \frac{d}{dt} F \rangle = -f\langle F, \nu \rangle$$

and by writing $F = DF(\nabla s) + \langle F, \nu \rangle \nu$

$$-\phi |\nabla s|^2 = -\phi(2s - \langle F, \nu \rangle^2).$$

The combination gives

$$\begin{aligned} \frac{d}{dt} s &= -f\langle F, \nu \rangle + \Delta s - n + f\langle F, \nu \rangle + \phi \langle F, \nu \rangle^2 \\ &- \phi |\nabla s|^2 + \phi(2s - \langle F, \nu \rangle^2) \\ &= \Delta s - \phi |\nabla s|^2 + 2s\phi - n. \end{aligned}$$

□

Next we want to compute the evolution of $\langle F, \nu \rangle$.

Lemma 4.3 *$\langle F, \nu \rangle$ satisfies*

$$\begin{aligned} \frac{d}{dt} \langle F, \nu \rangle &= \Delta \langle F, \nu \rangle - \phi \langle \nabla s, \nabla \langle F, \nu \rangle \rangle + \langle F, \nu \rangle (|A|^2 + \phi + \phi' \langle F, \nu \rangle^2) \\ &- 2H - 2s\phi' \langle F, \nu \rangle \end{aligned}$$

Proof: We compute

$$\nabla_i \langle F, \nu \rangle = h_i^{\ l} \nabla_l s$$

and

$$\nabla_i \nabla_j \langle F, \nu \rangle = \nabla^l h_{ij} \nabla_l s + h_{ij} - \langle F, \nu \rangle h_i^{\ l} h_{lj},$$

where we used the Codazzi equation (8). This gives

$$\Delta \langle F, \nu \rangle = \langle \nabla s, \nabla H \rangle + H - |A|^2 \langle F, \nu \rangle.$$

On the other hand with (13) and (18) we conclude

$$\frac{d}{dt} \langle F, \nu \rangle = -f + \langle F, DF(\nabla f) \rangle = -f + \langle \nabla s, \nabla f \rangle$$

so that

$$\frac{d}{dt} \langle F, \nu \rangle = \Delta \langle F, \nu \rangle - \langle \nabla s, \nabla (H - f) \rangle - H - f + |A|^2 \langle F, \nu \rangle.$$

Since

$$\nabla (H - f) = \langle F, \nu \rangle \phi' \nabla s + \phi \nabla \langle F, \nu \rangle \tag{21}$$

and

$$|\nabla s|^2 = 2s - \langle F, \nu \rangle^2$$

we obtain the result. □

We want to rewrite equations (15) and (16). To this end we need an expression for $\nabla_i \nabla_j (f - H)$. From (21) we obtain

$$\begin{aligned} \nabla_i \nabla_j (f - H) &= -\nabla_i (\langle F, \nu \rangle \phi' \nabla_j s + \phi \nabla_j \langle F, \nu \rangle) \\ &= -\nabla_i \big((\langle F, \nu \rangle \phi' \delta_j^{\ l} + \phi h_j^{\ l}) \nabla_l s \big) \\ &= -(h_i^{\ k} \nabla_k s \phi' \delta_j^{\ l} + \langle F, \nu \rangle \phi'' \nabla_i s \delta_j^{\ l} + \phi' \nabla_i s h_j^{\ l} + \phi \nabla^l h_{ij}) \nabla_l s \\ &- \ (\langle F, \nu \rangle \phi' \delta_j^{\ l} + \phi h_j^{\ l})(g_{il} - \langle F, \nu \rangle h_{il}) \\ &= -\phi' (h_i^{\ l} \nabla_j s + h_j^{\ l} \nabla_i s) \nabla_l s - \phi'' \langle F, \nu \rangle \nabla_i s \nabla_j s - \phi \nabla^l h_{ij} \nabla_l s \\ &- \ (\phi - \phi' \langle F, \nu \rangle^2) h_{ij} + \phi \langle F, \nu \rangle h_i^{\ l} h_{lj} - \phi' \langle F, \nu \rangle g_{ij}. \end{aligned} \tag{22}$$

Then

$$\begin{aligned} \Delta (f - H) &= -2\phi' h^{ij} \nabla_i s \nabla_j s - \phi'' \langle F, \nu \rangle (2s - \langle F, \nu \rangle^2) - \phi \langle \nabla s, \nabla H \rangle \\ &- \ (\phi - \phi' \langle F, \nu \rangle^2) H + \phi \langle F, \nu \rangle |A|^2 - n \phi' \langle F, \nu \rangle. \end{aligned}$$

Now (15) implies

Lemma 4.4

$$\begin{aligned}\frac{d}{dt}H &= \Delta H - \phi\langle\nabla s, \nabla H\rangle + H(|A|^2 - \phi + \phi'\langle F,\nu\rangle^2)\\ &- 2\phi' h^{ij}\nabla_i s\nabla_j s + \langle F,\nu\rangle(\phi''\langle F,\nu\rangle^2 - 2s\phi'' - n\phi').\end{aligned}$$

(22) also gives

$$\begin{aligned}2h^{ij}\nabla_i\nabla_j(f-H) &= -4\phi' h^{il}h_l^{\,j}\nabla_i s\nabla_j s - 2\phi''\langle F,\nu\rangle h^{ij}\nabla_i s\nabla_j s\\ &- \phi\langle\nabla s, \nabla|A|^2\rangle - 2|A|^2(\phi - \phi'\langle F,\nu\rangle^2)\\ &+ 2\phi\langle F,\nu\rangle C - 2\phi'\langle F,\nu\rangle H.\end{aligned}$$

From (16) and Simons' identity (10) we then derive

$$\begin{aligned}\frac{d}{dt}|A|^2 &= 2h^{ij}\nabla_i\nabla_j f + 2fC = 2h^{ij}\nabla_i\nabla_j H + 2h^{ij}\nabla_i\nabla_j(f-H) + 2fC\\ &= \Delta|A|^2 - 2|\nabla A|^2 - 2Z\\ &- 4\phi' h^{il}h_l^{\,j}\nabla_i s\nabla_j s - 2\phi''\langle F,\nu\rangle h^{ij}\nabla_i s\nabla_j s - \phi\langle\nabla s, \nabla|A|^2\rangle\\ &- 2|A|^2(\phi - \phi'\langle F,\nu\rangle^2) + 2\phi\langle F,\nu\rangle C - 2\phi'\langle F,\nu\rangle H + 2fC.\end{aligned}$$

Cancellation gives

Lemma 4.5

$$\begin{aligned}\frac{d}{dt}|A|^2 &= \Delta|A|^2 - \phi\langle\nabla s, \nabla|A|^2\rangle - 2|\nabla A|^2\\ &- 4\phi' h^{il}h_l^{\,j}\nabla_i s\nabla_j s - 2\phi''\langle F,\nu\rangle h^{ij}\nabla_i s\nabla_j s\\ &+ 2|A|^2(|A|^2 - \phi + \phi'\langle F,\nu\rangle^2) - 2\phi'\langle F,\nu\rangle H.\end{aligned}$$

In the next steps we want to prove that our spheres remain starshaped and that (4) remains true under the evolution. Therefore we define the quantity

$$q := \frac{f}{\langle F,\nu\rangle}.$$

A geometric motivation for q will be given in the appendix. Then the evolution equations for f and $\langle F,\nu\rangle$ imply

$$\begin{aligned}\frac{d}{dt}q &= \frac{1}{\langle F,\nu\rangle}\big(\Delta f - \phi\langle\nabla s, \nabla f\rangle + f(|A|^2 + \phi + \phi'\langle F,\nu\rangle^2)\big)\\ &- \frac{f}{\langle F,\nu\rangle^2}\big(\Delta\langle F,\nu\rangle - \phi\langle\nabla s, \nabla\langle F,\nu\rangle\rangle + \langle F,\nu\rangle(|A|^2 + \phi + \phi'\langle F,\nu\rangle^2)\\ &- 2H - 2s\phi'\langle F,\nu\rangle\big).\end{aligned} \tag{23}$$

On the other hand we compute

$$\begin{aligned}\Delta q &= \frac{\Delta f}{\langle F,\nu\rangle} + f\big(-\frac{1}{\langle F,\nu\rangle^2}\Delta\langle F,\nu\rangle + \frac{2}{\langle F,\nu\rangle^3}|\nabla\langle F,\nu\rangle|^2\big)\\ &- \frac{2}{\langle F,\nu\rangle^2}\langle\nabla f,\nabla\langle F,\nu\rangle\rangle\\ &= \frac{\Delta f}{\langle F,\nu\rangle} - \frac{f}{\langle F,\nu\rangle^2}\Delta\langle F,\nu\rangle - \frac{2}{\langle F,\nu\rangle}\langle\nabla\langle F,\nu\rangle,\nabla q\rangle\end{aligned}$$

Inserting this in (23) yields

Lemma 4.6

$$\frac{d}{dt}q = \Delta q + \frac{2}{\langle F,\nu\rangle}\langle\nabla\langle F,\nu\rangle,\nabla q\rangle - \phi\langle\nabla s,\nabla q\rangle + 2q(q+s\phi'+\phi). \tag{24}$$

Corollary 4.7 *Under the assumptions of Theorem 1 there exists a positive constant $\epsilon > 0$ independent of t such that*

$$q + s\phi' + \phi \le -\epsilon, \tag{25}$$

$$q^2 \le (\max_{t=0} q^2)e^{-4\epsilon t} \tag{26}$$

as long as a smooth starshaped solution of (1) exists.

Proof: From Lemma 4.6 we obtain

$$\begin{aligned}\frac{d}{dt}q^2 &= \Delta q^2 - 2|\nabla q|^2 + \frac{2}{\langle F,\nu\rangle}\langle\nabla\langle F,\nu\rangle,\nabla q^2\rangle - \phi\langle\nabla s,\nabla q^2\rangle\\ &+ 4q^2(q+s\phi'+\phi).\end{aligned} \tag{27}$$

Since $s\phi' + \phi = \frac{n}{2}\eta'$, the barrier argument, Lemma 3.3, and inequality (3) imply that

$$s\phi' + \phi < -c_\eta$$

as long as S_t remains smooth and starshaped. Consequently

$$\begin{aligned}\frac{d}{dt}q^2 &\le \Delta q^2 - 2|\nabla q|^2 + \frac{2}{\langle F,\nu\rangle}\langle\nabla\langle F,\nu\rangle,\nabla q^2\rangle - \phi\langle\nabla s,\nabla q^2\rangle\\ &+ 4q^2(q-c_\eta).\end{aligned} \tag{28}$$

Hence the maximum principle and (4) imply that q^2 cannot develop a positive maximum with $\frac{d}{dt}q^2 > 0$ and $q - c_\eta$ remains bounded above by its initial (negative) maximum. This proves (25) with

$$\epsilon := c_\eta - \max_{t=0} q.$$

But then the evolution equation for q^2 and (25) even imply that $q^2\, e^{4\epsilon t}$ remains bounded by its initial maximum.

□

Lemma 4.8 *Under the assumptions of Theorem 1 the starshapedness of S_t remains true as long as a smooth solution of (1) exists.*

Proof: We compute the evolution for the quantity

$$\frac{1}{\langle F, \nu\rangle^2}$$

which remains bounded above if and only if S_t remains starshaped. From Lemma 4.3 we get

$$\begin{aligned} \frac{d}{dt}\frac{1}{\langle F,\nu\rangle^2} &= -\frac{2}{\langle F,\nu\rangle^3}\big(\Delta\langle F,\nu\rangle - \phi\langle\nabla s, \nabla\langle F,\nu\rangle\rangle \\ &+ \langle F,\nu\rangle(|A|^2 + \phi + \phi'\langle F,\nu\rangle^2) - 2H - 2s\phi'\langle F,\nu\rangle\big) \\ &= \Delta\frac{1}{\langle F,\nu\rangle^2} - \frac{6}{\langle F,\nu\rangle^4}|\nabla\langle F,\nu\rangle|^2 - \phi\left\langle\nabla s, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle \\ &+ \frac{2}{\langle F,\nu\rangle^2}(-|A|^2 - \phi - \phi'\langle F,\nu\rangle^2 + 2(q + s\phi' + \phi)). \end{aligned} \tag{29}$$

Now as long as $\langle F,\nu\rangle$ remains positive we can use (25) and estimate

$$\begin{aligned} \frac{d}{dt}\frac{1}{\langle F,\nu\rangle^2} &\le \Delta\frac{1}{\langle F,\nu\rangle^2} - \phi\left\langle\nabla s, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle + \frac{2}{\langle F,\nu\rangle}\left\langle\nabla\langle F,\nu\rangle, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle \\ &+ \frac{2}{\langle F,\nu\rangle^2}(-|A|^2 - \phi - 2\epsilon) - 2\phi'. \end{aligned}$$

The barrier argument, Lemma 3.3, implies

$$\phi(s(x,t)) \geq \min_{\sigma\in\left[\min\left\{\min\limits_{t=t_0}\frac{|F|^2}{2},s_0\right\},\max\left\{\max\limits_{t=t_0}\frac{|F|^2}{2},s_0\right\}\right]}\phi(\sigma) =: c_{\phi,t_0}, \tag{30}$$

$$\phi'(s(x,t)) \geq \min_{\sigma\in\left[\min\left\{\min\limits_{t=t_0}\frac{|F|^2}{2},s_0\right\},\max\left\{\max\limits_{t=t_0}\frac{|F|^2}{2},s_0\right\}\right]}\phi'(\sigma) =: c_{\phi',t_0} \tag{31}$$

for all $x \in S^n$ and all $t \geq t_0$, where $t_0 \geq 0$ is a fixed time. For the proof of this lemma we will actually only use $t_0 = 0$ but we need the same estimates for a different t_0 later. Hence for all $t \geq t_0$ the following inequality is valid for starshaped S_t

$$\begin{aligned}\frac{d}{dt}\frac{1}{\langle F,\nu\rangle^2} &\leq \Delta\frac{1}{\langle F,\nu\rangle^2} - \phi\left\langle \nabla s, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle + \frac{2}{\langle F,\nu\rangle}\left\langle \nabla\langle F,\nu\rangle, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle \\ &- \frac{2(|A|^2+2\epsilon+c_{\phi,t_0})}{\langle F,\nu\rangle^2} - 2c_{\phi',t_0}.\end{aligned} \tag{32}$$

The last term in the first line is nonpositive but we keep it for later purposes. Let $p(t)$ be the solution of

$$\begin{aligned}\frac{d}{dt}p &= -2(2\epsilon + c_{\phi,t_0})p - 2c_{\phi',t_0}, \\ p(t_0) &= \max_{t=t_0}\frac{1}{\langle F,\nu\rangle^2},\end{aligned}$$

i. e. for $2\epsilon + c_{\phi,t_0} \neq 0$

$$p(t) = \left(p(t_0) + \frac{c_{\phi',t_0}}{2\epsilon + c_{\phi,t_0}}\right)e^{-2(2\epsilon+c_{\phi,t_0})(t-t_0)} - \frac{c_{\phi',t_0}}{2\epsilon + c_{\phi,t_0}}.$$

Then for all $t \geq t_0$ and starshaped S_t

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{\langle F,\nu\rangle^2} - p\right) &\leq \Delta\left(\frac{1}{\langle F,\nu\rangle^2} - p\right) - \phi\left\langle\nabla s, \nabla\left(\frac{1}{\langle F,\nu\rangle^2} - p\right)\right\rangle \\ &- 2(2\epsilon + c_{\phi,t_0})\left(\frac{1}{\langle F,\nu\rangle^2} - p\right).\end{aligned}$$

The maximum principle then implies

$$\frac{1}{\langle F,\nu\rangle^2} \leq \left(\max_{t=t_0}\frac{1}{\langle F,\nu\rangle^2} + \frac{c_{\phi',t_0}}{2\epsilon + c_{\phi,t_0}}\right)e^{-2(2\epsilon+c_{\phi,t_0})(t-t_0)} - \frac{c_{\phi',t_0}}{2\epsilon + c_{\phi,t_0}} \tag{33}$$

for all $t \geq t_0$. Consequently the quantity $\langle F, \nu \rangle$ cannot tend to zero in finite time and S_t remains starshaped. $\square$

Theorem 4.9 *With the same assumptions as in Theorem 1 a smooth solution exists for all $t > 0$.*

Proof: We need bounds for $|\nabla^k A|^2$ for all $k \geq 0$. We begin with a bound for

$$\frac{|A|^2}{\langle F, \nu \rangle^2}.$$

From Lemma 4.5 and the evolution equation (29) we obtain

$$\begin{aligned}
\frac{d}{dt}\frac{|A|^2}{\langle F,\nu\rangle^2} &= |A|^2\big(\Delta\frac{1}{\langle F,\nu\rangle^2} - \frac{6}{\langle F,\nu\rangle^4}|\nabla\langle F,\nu\rangle|^2 - \phi\left\langle \nabla s, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle \\
&+ \frac{2}{\langle F,\nu\rangle^2}(-|A|^2 - \phi - \phi'\langle F,\nu\rangle^2 + 2(q + s\phi' + \phi))\big) \\
&+ \frac{1}{\langle F,\nu\rangle^2}\big(\Delta|A|^2 - \phi\langle\nabla s, \nabla|A|^2\rangle - 2|\nabla A|^2 \\
&- 4\phi' h^{il}h_l^{\ j}\nabla_i s\nabla_j s - 2\phi''\langle F,\nu\rangle h^{ij}\nabla_i s\nabla_j s \\
&+ 2|A|^2(|A|^2 - \phi + \phi'\langle F,\nu\rangle^2) - 2\phi'\langle F,\nu\rangle H\big) \\
&= \Delta\frac{|A|^2}{\langle F,\nu\rangle^2} - 2\left\langle \nabla|A|^2, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle - \phi\left\langle\nabla s, \nabla\frac{|A|^2}{\langle F,\nu\rangle^2}\right\rangle \\
&- 2\frac{|\nabla A|^2}{\langle F,\nu\rangle^2} - 6\frac{|A|^2|\nabla\langle F,\nu\rangle|^2}{\langle F,\nu\rangle^4} - 4\phi'\frac{|\nabla\langle F,\nu\rangle|^2}{\langle F,\nu\rangle^2} \\
&- 2\frac{\phi''}{\langle F,\nu\rangle}h^{ij}\nabla_i s\nabla_j s - 2\phi'\frac{H}{\langle F,\nu\rangle} + 4\frac{|A|^2}{\langle F,\nu\rangle^2}(q + s\phi').
\end{aligned}$$

To proceed we define

$$\begin{aligned}
Q^2 &:= |\langle F,\nu\rangle\nabla_i h_{jk} - \nabla_i\langle F,\nu\rangle h_{jk}|^2 \\
&= \langle F,\nu\rangle^2|\nabla A|^2 + |A|^2|\nabla\langle F,\nu\rangle|^2 - \langle F,\nu\rangle\langle\nabla\langle F,\nu\rangle, \nabla|A|^2\rangle.
\end{aligned}$$

This implies

$$\begin{aligned}
&- 2\frac{|\nabla A|^2}{\langle F,\nu\rangle^2} - 6\frac{|A|^2|\nabla\langle F,\nu\rangle|^2}{\langle F,\nu\rangle^4} - 2\left\langle\nabla|A|^2, \nabla\frac{1}{\langle F,\nu\rangle^2}\right\rangle \\
&= -2\frac{Q^2}{\langle F,\nu\rangle^4} + \frac{2}{\langle F,\nu\rangle}\left\langle\nabla\langle F,\nu\rangle, \nabla\frac{|A|^2}{\langle F,\nu\rangle^2}\right\rangle
\end{aligned}$$

and finally

$$
\begin{aligned}
\frac{d}{dt}\frac{|A|^2}{\langle F,\nu\rangle^2} &= \Delta\frac{|A|^2}{\langle F,\nu\rangle^2} - \phi\left\langle \nabla s, \nabla\frac{|A|^2}{\langle F,\nu\rangle^2}\right\rangle + \frac{2}{\langle F,\nu\rangle}\left\langle \nabla\langle F,\nu\rangle, \nabla\frac{|A|^2}{\langle F,\nu\rangle^2}\right\rangle \\
&- 2\frac{Q^2}{\langle F,\nu\rangle^4} - 4\phi'\frac{|\nabla\langle F,\nu\rangle|^2}{\langle F,\nu\rangle^2} - 2\frac{\phi''}{\langle F,\nu\rangle}h^{ij}\nabla_i s\nabla_j s \\
&- 2\phi'\frac{H}{\langle F,\nu\rangle} + 4\frac{|A|^2}{\langle F,\nu\rangle^2}(q+s\phi'). \qquad (34)
\end{aligned}
$$

To proceed we need the estimate

$$
\begin{aligned}
|\nabla\langle F,\nu\rangle|^2 &= h^i{}_l h^{lj}\nabla_i s\nabla_j s \\
&\le |A|^2|\nabla s|^2 \\
&\le 2s|A|^2.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\left|\frac{H}{\langle F,\nu\rangle}\right| &\le \frac{H^2}{2\langle F,\nu\rangle^2} + \frac{1}{2} \\
&\le \frac{n|A|^2}{2\langle F,\nu\rangle^2} + \frac{1}{2}.
\end{aligned}
$$

Then we use the barrier argument, Lemma 3.3, inequality (25) and estimate

$$
\begin{aligned}
\frac{d}{dt}\frac{|A|^2}{\langle F,\nu\rangle^2} &\le \Delta\frac{|A|^2}{\langle F,\nu\rangle^2} - \phi\left\langle \nabla s, \nabla\frac{|A|^2}{\langle F,\nu\rangle^2}\right\rangle + \frac{2}{\langle F,\nu\rangle}\left\langle \nabla\langle F,\nu\rangle, \nabla\frac{|A|^2}{\langle F,\nu\rangle^2}\right\rangle \\
&+ c_1\frac{|A|^2}{\langle F,\nu\rangle^2} + c_2 \qquad (35)
\end{aligned}
$$

for two positive constants c_1, c_2 independent of t. So we see that $\frac{|A|^2}{\langle F,\nu\rangle^2}$ can increase at most exponentially in time and since by Lemma 4.8 S_t remains starshaped and $|\langle F,\nu\rangle| \le \text{const}$, this also means that $|A|^2$ remains bounded on any finite time interval. We can then proceed as in [2] to derive uniform upper bounds for all higher covariant derivatives of A on any finite time interval $[0,\widetilde{T})$ for which a smooth solution of the flow exists. Consequently $T=\infty$. □

Proof of the main theorem: We are now able to prove uniform upper bounds in t also. First we need a uniform upper bound for $\frac{1}{\langle F,\nu\rangle^2}$. Therefore we go back to inequality (33). From Lemma 4.9 we know that the flow exists for all $t > 0$ and since by assumption (2) $\phi(s_0) = \frac{n\eta(s_0)}{2s_0} = \frac{n}{2s_0} > 0$, the barrier argument, Lemma 3.3, implies the existence of $t_0 \geq 0$ estimated from above such that $c_{\phi,t_0} > 0$. But then inequality (33) implies that with a constant $c_3 > 0$ and for all $t > 0$

$$\frac{1}{\langle F, \nu\rangle^2} \leq c_3. \tag{36}$$

Our idea is to add enough of $\frac{1}{\langle F,\nu\rangle^2}$ to $\frac{|A|^2}{\langle F,\nu\rangle^2}$ to obtain a uniform bound for $|A|^2$. Therefore, let

$$B := \frac{|A|^2 + k}{\langle F, \nu\rangle^2}$$

for a large constant k to be determined. Then (32) and (35) give

$$\begin{aligned} \frac{d}{dt}B &\leq \Delta B - \phi\langle \nabla s, \nabla B\rangle + \frac{2}{\langle F,\nu\rangle}\langle \nabla\langle F,\nu\rangle, \nabla B\rangle \\ &+ (c_1 - 2k)\frac{|A|^2}{\langle F,\nu\rangle^2} - k\left(\frac{2(2\epsilon + c_{\phi,t_0})}{\langle F,\nu\rangle^2} + 2c_{\phi',t_0}\right) + c_2 \end{aligned} \tag{37}$$

for all $t \geq t_0$, with a fixed t_0, without loss of generality $t_0 = 0$. We choose $k = c_1$. Then

$$\begin{aligned} \frac{d}{dt}B &\leq \Delta B - \phi\langle \nabla s, \nabla B\rangle + \frac{2}{\langle F,\nu\rangle}\langle \nabla\langle F,\nu\rangle, \nabla B\rangle \\ &- c_1 B + c_4, \end{aligned} \tag{38}$$

where c_4 depends only on $c_1, c_2, c_3, \epsilon, c_{\phi,0}$ and $c_{\phi',0}$ but not on t. Consequently a maximum of B with $\frac{d}{dt}B > 0$ must be smaller than $\frac{c_4}{c_1}$ and furthermore B and $|A|^2$ are uniformly bounded. Once we've obtained a uniform bound for $|A|^2$ we use the technique in [2] to obtain uniform upper bounds for all quantities $|\nabla^k A|^2$. This shows that F_t is uniformly bounded in C^∞ for all $t \geq 0$. From the barrier argument, Lemma 3.3, we conclude that $|s - s_0|$ decays exponentially. Then we use the elementary interpolation inequality

$$||\nabla\psi||^2 \leq \text{const}\big(\widetilde{M}, \widetilde{g}\big) \cdot ||\psi|| \cdot \big(||\nabla\psi|| + ||\nabla^2\psi||\big),$$

for C^2-functions ψ on a compact Riemannian manifold $(\widetilde{M}, \widetilde{g})$, where ∇ denotes the covariant derivative w. r. t. $\widetilde{g}$ and the constant depends only on

$\widetilde{M}$ and $\widetilde{g}$. This inequality, the C^k-bounds for $s - s_0$ (which follow from those for the second fundamental form) and the uniform equivalence of the induced metrics on $M = S^n$ for any t imply an exponential decay of $|\nabla^k(s - s_0)|^2$ for all $k \in \mathbb{N}$. We thus conclude exponential convergence in C^∞ to the stable sphere of radius $r_0 = \sqrt{2s_0}$. □

5 Appendix

Corollary 5.1 *Assume $f > 0$ for $t = 0$. If the absolute value of $|A|^2 + \phi + \phi'\langle F, \nu\rangle^2$ remains bounded from above by some positive constant c on the time interval $[0, t_0), t_0 \leq T$, then for $t \in [0, t_0)$*

$$\min_{x \in M_t} f(x) \geq \min_{x \in M_0} f(x) e^{-ct} > 0.$$

Proof: This is a direct consequence of the parabolic maximum principle applied to Lemma 4.1. □

A geometric motivation for $\frac{f}{\langle F, \nu \rangle}$

We wish to represent our evolution problem via graphs over the sphere S^n as in [1]. Therefore we use an embedding of the form

$$F = x\left(\xi^i, t\right) \cdot u\left(x\left(\xi^i\right), t\right),$$

where $x \in S^n$ and ξ^i denotes local coordinates of S^n. We use covariant derivatives with respect to the metric of S^n and compute

$$
\begin{aligned}
F_i &= x_i u + x u_i, \\
F_{ij} &= x_{ij} u + x_i u_j + x_j u_i + x u_{ij}, \\
g_{ij} &= \langle F_i, F_j \rangle = u^2 \sigma_{ij} + u_i u_j \\
&= u^2 (\sigma_{ij} + \varphi_i \varphi_j), \qquad \text{where } \varphi = \log u, \\
g^{ij} &= u^{-2} \left(\sigma^{ij} - \frac{\varphi^i \varphi^j}{w^2} \right), \\
\varphi^k &= \sigma^{kl} \varphi_l, \\
w &:= \sqrt{1 + \varphi^i \varphi_i},
\end{aligned}
$$

where σ_{ij} denotes the metric of the sphere S^n. The outer unit normal is given by

$$
\nu = \frac{1}{w} \left(x - \varphi^k x_k \right).
$$

The Gauß formula relates the second covariant derivatives of the embedding vector to the second fundamental form and the normal, where the second derivatives are taken with respect to the induced metric. At the moment, however, F_{ij} denotes the second covariant derivatives with respect to the metric σ_{ij}. However, these derivatives are related by

$$
F^g_{ij} = F^\sigma_{ij} + \left(\Gamma^k_{ij}(\sigma) - \Gamma^k_{ij}(g) \right) F_k,
$$

so we deduce from the Gauß formulae for M and S^n

$$
\begin{aligned}
\langle F^\sigma_{ij}, \nu \rangle &= \langle -h_{ij} \nu, \nu \rangle = -h_{ij}, \\
h_{ij} &= -\frac{1}{w} \left\langle x - \frac{1}{u} u^k x_k, u_{ij} x + u_i x_j + u_j x_i - u \sigma_{ij} x \right\rangle \\
&= \frac{u}{w} \left(\sigma_{ij} + 2 \frac{1}{u^2} u_i u_j - \frac{1}{u} u_{ij} \right) \\
&= \frac{u}{w} (\sigma_{ij} + \varphi_i \varphi_j - \varphi_{ij}) \\
&= \frac{1}{uw} g_{ij} - \frac{u}{w} \varphi_{ij}.
\end{aligned}
$$

Next, we will derive an evolution equation for u

$$\begin{aligned} (39) \qquad \frac{d}{dt}(x\cdot u) &= \frac{\partial}{\partial t}x\cdot u + x\cdot\frac{\partial}{\partial t}u + x\cdot\left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle, \\ \frac{d}{dt}F &= -f\nu = -f\frac{1}{w}\left(x - \varphi^i x_i\right). \end{aligned}$$

In view of

$$0 = \frac{\partial}{\partial t}\langle x, x\rangle = 2\left\langle \frac{\partial}{\partial t}x, x\right\rangle,$$

we obtain by multiplying (39) with x

$$-f\frac{1}{w} = \frac{\partial u}{\partial t} + \left\langle \nabla u, \frac{\partial x}{\partial t}\right\rangle.$$

From the remaining part of (39) we get

$$f\frac{1}{w}\varphi^i x_i = \frac{\partial}{\partial t}x\cdot u,$$

thus

$$\begin{aligned} \frac{\partial}{\partial t}u &= -f\frac{1}{w}\left(1 + \left\langle \nabla\varphi, \varphi^i x_i\right\rangle\right) \\ &= -fw \end{aligned}$$

and in view of $\langle F, \nu\rangle = \frac{u}{w}$

$$\dot\varphi := \frac{\partial}{\partial t}\varphi = -\frac{f}{\langle F, \nu\rangle}.$$

We now state the parabolic differential equations for φ, $\dot\varphi$ and $W = \frac{1}{2}(1 + \varphi^k\varphi_k)$ to give the reader the opportunity to compare the evolution equations for $\dot\varphi$ and q, Lemma 4.6, as well as those for W and $\frac{1}{\langle F,\nu\rangle^2}$, equation (29),

$$\begin{aligned} \dot\varphi &= g^{ij}\varphi_{ij} - ne^{-2\varphi} + \phi, \\ \ddot\varphi &= g^{ij}\dot\varphi_{ij} - e^{-2\varphi}\frac{1}{w^2}(\dot\varphi^i\varphi^j + \varphi^i\dot\varphi^j)\varphi_{ij} + 2e^{-2\varphi}\frac{1}{w^4}\varphi^i\varphi^j\varphi_{ij}\varphi^k\dot\varphi_k \\ &\quad + \dot\varphi(-2\dot\varphi + 2\phi + \phi' e^{2\varphi}), \\ \dot W &= g^{ij}W_{ij} - e^{-2\varphi}\varphi_i^j\varphi_j^i - \frac{1}{w^2}e^{-2\varphi}W^iW_i + 2\frac{1}{w^4}e^{-2\varphi}(W^i\varphi_i)^2 \\ &\quad + (2W-1)\left(-e^{-2\varphi}(n-1) - 2\dot\varphi + 2\phi + \phi' e^{2\varphi}\right). \end{aligned}$$

References

[1] C. Gerhardt: Flow of nonconvex hypersurfaces into spheres. J. Differential Geom. 32 (1990), 299–314.

[2] G. Huisken: Flow by mean curvature of convex surfaces into spheres. J. Differential Geom. **20** (1984), 237–266.

[3] G. Huisken, A. Polden: Geometric evolution equations for hypersurfaces. Calculus of variations and geometric evolution problems (Cetraro, 1996), 45–84, Lecture Notes in Math., **1713**, Springer, Berlin, 1999.

[4] N. M. Ivochkina, Th. Nehring, F. Tomi: Evolution of starshaped hypersurfaces by nonhomogeneous curvature functions. Algebra i Analiz **12** (2000), 185–203.

[5] K. Smoczyk: Symmetric hypersurfaces in Riemannian manifolds contracting to Lie-groups by their mean curvature. Calc. Var. Partial Differential Equations **4** (1996), 155–170.

[6] K. Smoczyk: Starshaped hypersurfaces and the mean curvature flow. Manuscripta Math. **95** (1998), 225–236.

[7] J. I. E. Urbas: On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z. **205** (1990), 355–372.

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