# ANGLE THEOREMS FOR THE LAGRANGIAN MEAN CURVATURE FLOW

KNUT SMOCZYK

ABSTRACT. We prove that symplectic maps between Riemann surfaces $L$, $M$ of constant, nonpositive and equal curvature converge to minimal symplectic maps, if the Lagrangian angle $\alpha$ for the corresponding Lagrangian submanifold in the cross product space $L \times M$ satisfies $\text{osc}(\alpha) \leq \pi$. If one considers a 4-dimensional Kähler-Einstein manifold $\overline{M}$ of nonpositive scalar curvature that admits two complex structures $J, K$ which commute and assumes that $L \subset \overline{M}$ is a compact oriented Lagrangian submanifold w.r.t. $J$ such that the Kähler form $\overline{\kappa}$ w.r.t. $K$ restricted to $L$ is positive and $\text{osc}(\alpha) \leq \pi$, then $L$ converges under the mean curvature flow to a minimal Lagrangian submanifold which is calibrated w.r.t. $\overline{\kappa}$.

## 1. INTRODUCTION

In symplectic geometry there is a destinguished class of immersions, known as Lagrangian submanifolds. These are $n$-dimensional submanifolds $L$ in $2n$-dimensional symplectic manifolds $(M, \overline{\omega})$ such that $\omega := \overline{\omega}_{TL} = 0$. They are important in physics and of course in pure and applied mathematics as well. E. g., if $\overline{M}$ is a Calabi-Yau 3-fold, then $H^{1,1}(\overline{M})$ and $H^{2,1}(\overline{M})$ can be recovered from associated Conformal Field Theories (CFT) as eigenspaces of a certain operator (for more details on mirror symmetry see [11]). The only difference between the CFT representations for $H^{1,1}(\overline{M})$ and $H^{2,1}(\overline{M})$ is the sign of their eigenvalue under a $U(1)$-action. Since the sign is only a matter of convention, this led some physicists to conjecture that there should exist a Calabi-Yau 3-fold $\widetilde{M}$ with the same CFT but with different signs for the operators, so that $H^{1,1}(\overline{M}) = H^{2,1}(\widetilde{M})$ and the mirror conjecture was born. More recently, Strominger Yau and Zaslow [7] conjectured that a fibration of a Calabi-Yau manifold by special Lagrangian tori (these are Lagrangian tori calibrated w.r.t. the real part of the complex volume form and consequently they are minimal) can be used to construct the mirror manifold. However in general some of these tori will be singular. To understand the nature of minimal Lagrangian submanifolds in a Calabi-Yau manifold or more generally in a Kähler-Einstein manifold it would be very useful to obtain a general mechanism to produce such minimal immersions.

---

*Date*: March 2001.

This work has been carried out at the Max-Planck-Institute for Mathematics in the Sciences, Leipzig and the author would like to thank Jürgen Jost for his support.

Since minimal submanifolds locally minimize volume and the mean curvature flow is the negative gradient flow of the volume energy one is led to the mean curvature flow for Lagrangian submanifolds. There is, however, one restriction. In order to preserve the Lagrangian nature an integrability condition has to be imposed. To make this precise, let us recall that a normal vector field $V \in NL$ on a Lagrangian submanifold can be identified with a 1-form $\theta_V$ on $L$ via $\overline{\omega}(X, V) = \theta_V(X), \quad \forall X \in TL$, where $\overline{\omega}$ is the Kähler form of the ambient manifold. $V$ and $\theta_V$ are then said to be associated. The associated 1-form $H$ of the mean curvature vector field $\overrightarrow{H}$ is called mean curvature form. The integrability condition now states that $dH = 0$ which is true in Kähler-Einstein manifolds and this implies the following well known short-time existence result:

**Proposition 1.1.** *Let $L$ be a compact manifold and let $F_0 : L \to L_0 \subset M$ be a smooth immersion of $L$ as a Lagrangian submanifold into a Kähler-Einstein manifold $M$. Then the mean curvature flow*

$$\begin{aligned} \frac{d}{dt}F(x,t) &= \overrightarrow{H}(x,t) \\ F(x,0) &= F_0(x) \end{aligned} \tag{1.1}$$

*admits a smooth solution on some time interval $[0, T), \quad T > 0$ and the submanifolds $L_t := F_t(L)$ are all Lagrangian. Here $F_t$ is shorthand for $F_t(x) := F(x,t)$ and $\overrightarrow{H}$ is the mean curvature vector field of the submanifold $L_t$.*

In addition, if $\overline{R}_{\alpha\beta} = K\overline{g}_{\alpha\beta}$, then the 1-form $e^{-Kt}H$ does not change its cohomology class (compare with the evolution equation in Lemma 3.3, where $K = \frac{S}{n}$). So if $K \geq 0$ and the flow exists for all times converging to a minimal Lagrangian submanifold, we must have $H = d\alpha$ at $t = 0$ for some angle function $\alpha$, called the Lagrangian angle. However, $[H] = 0$ is not necessary in the case $K < 0$ as follows from simple examples of closed curves with constant curvature converging to closed geodesics on surfaces with negative curvature. The question, whether the condition $H = d\alpha$ is sufficient also, is easily answered with no. An example can be given by the Whitney sphere. These are the immersions of $S^n$ into $\mathbb{R}^{2n}$ defined by

$$f(x_0, x_1, \ldots, x_n) := (x_1, \ldots, x_n, x_0x_1, \ldots, x_0x_n).$$

Clearly, the mean curvature form is exact but there do not exist compact minimal (Lagrangian) immersions in the euclidean space. But still many people, including the author, believed that some condition on the mean curvature form $H$ should guarantee longtime existence and convergence to minimal Lagrangian immersions, at least in the case $K \leq 0$ which is more stable than the positive case. Before we state our main result let us commence with the following general results that will also be proven in this paper:

**Theorem 1.2.** *Let $L$ be a compact manifold and let $F_0 : L \to L_0 \subset \overline{M}$ be a smooth immersion of $L$ as a Lagrangian submanifold into a Kähler-Einstein manifold $\overline{M}$ that is either compact or complete with bounded curvature quantities. Further assume that the Lagrangian mean curvature flow admits a smooth solution on a maximal time interval $[0,T)$, $0 < T \le \infty$. Then the following is true:*

(a) *Assume there exists a constant $C_0 < \infty$ such that*

$$\max_{L_t} |A|^2 \le C_0, \quad \forall\, t \in [0,T),$$

*where $|A|^2$ is the squared norm of the second fundamental tensor $A$. Then for any $m \ge 0$ there exists a constant $C_m < \infty$ depending on $m, L_0, \overline{M}$ such that*

$$\max_{L_t} |\nabla^m A|^2 \le C_m, \quad \forall\, t \in [0,T).$$

(b) *If $T < \infty$, then*

$$\lim_{t\to T} \max_{L_t} |A|^2 = \infty.$$

(c) *If in addition to (a) the initial mean curvature form of $L_0$ is exact, the ambient Kähler-Einstein manifold has nonpositive Ricci curvature and the induced Riemannian metrics $g_{ij}(x,t)$ on $L$ are all uniformly equivalent, then $T = \infty$ and the Lagrangian submanifolds $L_t$ converge smoothly and exponentially to a smooth compact minimal Lagrangian immersion $L_\infty \subset M$.*

**Theorem 1.3.** *(1st angle theorem): Let $L_0$ be a compact, oriented Lagrangian submanifold in a Kähler-Einstein manifold $\overline{M}$. Assume that the mean curvature form of $L_0$ is exact and that the Lagrangian angle $\alpha$ with $d\alpha = H$ satisfies osc$(\alpha) \le \pi$. Then we have:*

(a) *If $\overline{M}$ is Ricci-flat, i.e. a Calabi-Yau manifold, then there exists a constant $c > 0$ depending only on osc$(\alpha)$ and Vol $(L_0)$ such that*

$$\text{Vol}\ (L_1) \ge c > 0$$

*for any Lagrangian submanifold $L_1 \subset \overline{M}$ which is Lagrangian isotopic to $L_0$.*

(b) *If the Ricci curvature of $\overline{M}$ is nonpositive (and constant in the case dim$(\overline{M}) = 2$), then the same result as in (a) holds for Lagrangian submanifolds $L_1$ which are Lagrangian isotopic to $L_0$ by means of the mean curvature flow.*

*Remark* 1.4. Theorem 1.3 is optimal, because for any $\epsilon > 0$ one can find a Lagrangian immersion of $S^1$ into $\mathbb{C}$ such that $\pi < \text{osc}(\alpha) < \pi + \epsilon$ and the homotheties in $\mathbb{C}$ are Lagrangian deformations. So the volume cannot be bounded from below by a universal positive constant in these cases.

*Remark* 1.5. The following can be seen easily and a short proof will be given in the appendix: If $L$ is a closed curve on a Riemann surface of constant nonpositive curvature, then the mean curvature flow deforms $L$ into a smooth closed geodesic, if $H = d\alpha$ with $\operatorname{osc}(\alpha) \leq \pi$. It follows from remark 1.4 that this result is optimal in the sense that one can find counterexamples in the case $\operatorname{osc}(\alpha) > \pi + \epsilon$ for any positive $\epsilon$.

Motivated by the last two theorems and Remark 1.5 we formulate the following conjecture:

*Conjecture* 1.6. Assume that $L$ is a closed Lagrangian immersion in a compact (or complete with bounded curvature quantities) Kähler-Einstein manifold $\overline{M}$ of nonpositive curvature such that $H = d\alpha$ with $\operatorname{osc}(\alpha) \leq \pi$. Then the mean curvature flow admits a smooth solution for all $t > 0$, smoothly converging to a minimal Lagrangian immersion in $\overline{M}$.

One of the standard examples for a Lagrangian submanifold is the graph of a symplectic map $f$. If $(M_1, \omega_1)$ and $(M_2, \omega_2)$ are two symplectic manifolds, then a smooth map $f : M_1 \to M_2$ is called symplectic, if $f^*\omega_2 = \omega_1$. Then a differentiable map $f : M_1 \to M_2$ is symplectic if and only if the graph

$$\Gamma_f := \{(x, f(x)) | x \in M_1\}$$

of $f$ is Lagrangian in the symplectic manifold $(M_1 \times M_2, (\omega_1, -\omega_2))$.

We note that Wolfson [10] studied symplectic maps on the unit disk with given boundary data. Our methods seem to apply also for domains with boundary under suitable boundary conditions.

If $f$ is a symplectic map such that the corresponding Lagrangian submanifold is minimal, then $f$ is called a minimal Lagrangian diffeomorphism or a symplectominimal map.

*Remark* 1.7. The condition $\operatorname{osc}(\alpha)$ can be interpreted as follows, at least in the case where $\overline{M}$ is Calabi-Yau: The real and imaginary parts of the complex volume form on $\overline{M}$ are calibrations and Lagrangian submanifolds that are calibrated w.r.t. the real part are called special Lagrangian. In particular these submanifolds are minimal. Now, the restriction of the complex volume form to the determinant bundle of an arbitrary Lagrangian submanifold yields a complex function $e^{i\alpha}$, with $d\alpha = H$ (note that $\alpha$ is in general a multivalued function). Therefore the real part of the complex volume form consists of $\cos(\alpha)$ and if the real part restricted to $L$ is positive, then $cos(\alpha) > 0$ so that $[H] = 0$ and $\operatorname{osc}(\alpha) \leq \pi$.

In this article the main result states that conjecture 1.6 is true for symplectic maps between Riemann surfaces of constant nonpositive curvature (the curvature must be constant to guarantee that the product metric is Einstein, so that the mean curvature flow preserves the Lagrangian condition). More precisely

**Theorem 1.8.** *(2nd angle theorem): Let $(L, J, \sigma)$ and $(M, K, \eta)$ be two Riemann surfaces with the same constant nonnegative scalar curvature, $L$ being* *compact. Assume*

$$f_0 : L \to M$$

*is a symplectic map such that the mean curvature form $H$ of the corresponding Lagrangian graph in*

$$(\overline{M} := L \times M, \overline{J} := (J, -K), \overline{g} := \sigma \times \eta)$$

*satisfies $H = d\alpha$ with $osc(\alpha) \leq \pi$. Then conjecture 1 is true and the Lagrangian surfaces smoothly and exponentially converge to a minimal Lagrangian surface and the symplectic map $f_0$ is Lagrangian isotopic to a symplectominimal map.*

*Remark* 1.9. Assume that $\overline{M}$ is a Kähler-Einstein 4-manifold of nonpositive scalar curvature that admits two complex structure $J, K$ which commute. Locally such manifolds $\overline{M}$ are product spaces. Moreover let $L$ be a compact surface in $\overline{M}$ which is Lagrangian w.r.t. $J$ and which satisfies the condition $\overline{\kappa}_{TL} > 0$, where $\overline{\kappa}$ denotes the Kähler form w.r.t. $K$. If $H = d\alpha$ with $\text{osc}(\alpha) \leq \pi$, then the same result holds as in theorem 1.8 because we can then apply the same proof as in Theorem 1.8 to bound all curvature quantities in the same way. In section 3 we compute the evolution equation for the function $r$ determined by $\overline{\kappa}_{|TL} = r\mu$, where $\mu$ is the volume form of $L$, see also remark 6.6 at the end of section 6.

The organization of the paper is as follows. In section 2 we fix the notation and recall basic equations in Kähler-Einstein manifolds and of their Lagrangian submanifolds. In sections 3, 4 we derive some evolution equations and prove Theorems 1.2 and 1.3. Then in the next section (section 5) we discuss the situation if the Lagrangian submanifolds are induced by symplectic maps. In section 6 we focus on the 2-dimensional case and will prove the main result which follows from rather complicated estimates for various curvature quantities. There we will also explain why, at the moment, we have to restrict to 2 dimensions. The paper ends with an appendix concerning the proof of Remark 1.5.

After this paper had been finished we learned that Mu-Tao Wang [9] obtained similar results in the case of positive curvature. In a paper by Thomas and Yau [8] some conjectures concerning the Lagrangian mean curvature flow have been made also.

## 2. Preliminaries and notations

### 2.1. Kähler manifolds.

We recall some well known identities for a Kähler manifold $(\overline{M}, \overline{J}, \overline{g})$. To this end assume that $(z^A)_{A=1,\ldots,2n}$ are local coordinates for $\overline{M}$, that doubled latin capitals are summed from 1 to $2n = \dim_{\mathbb{R}} \overline{M}$ and that an underlined minuscle denotes the application of the complex structure $\overline{J}$, e.g.

$$\overline{R}_{ABC\underline{D}} = \overline{R}(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}, \frac{\partial}{\partial z^C}, \overline{J}\frac{\partial}{\partial z^D}).$$

We let $\overline{R}_{AB}$ denote the Ricci curvature, $\overline{g}_{AB}$ the Riemannian metric and $\overline{\omega}_{AB}$ the Kähler form on $\overline{M}$. Let us also denote the Christoffel symbols for the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ by $\overline{\Gamma}^A_{BC}$ and we set

$$\overline{\Gamma}_{AB} := \overline{\Gamma}^C_{AB}\frac{\partial}{\partial z^C}.$$

Then:

**Proposition 2.1.** *The following relations hold on a Kähler manifold*

$$\begin{aligned}
\overline{R}_{ABCD} &= -\overline{R}_{ABDC} = \overline{R}_{BADC} = \overline{R}_{DCBA},\\
\textbf{1. Bianchi identity} &: \overline{R}_{ABCD} + \overline{R}_{ACDB} + \overline{R}_{ADBC} = 0, \qquad (2.1)\\
\textbf{2. Bianchi identity} &: \overline{\nabla}_A\overline{R}_{BCDE} + \overline{\nabla}_B\overline{R}_{CADE} + \overline{\nabla}_C\overline{R}_{ABDE} = 0,\\
\textbf{Compatibility} &: \overline{\omega}_{AB} = \overline{J}_A{}^C\overline{g}_{CB} = -\overline{J}_B{}^C\overline{g}_{CA} = -\overline{\omega}_{BA}.\\
\textbf{Kähler identity} &: \overline{R}_{ABC\underline{D}} = \overline{R}_{ABD\underline{C}}, \qquad (2.2)\\
2\overline{R}_{AB} &= \overline{R}_{\underline{A}B\underline{C}}{}^C, \qquad (2.3)\\
\overline{J}_A{}^B\overline{J}_C{}^A &= -\delta_C{}^B,\\
\overline{\nabla}_A\overline{J}_B{}^C &= 0,\\
\overline{\nabla}_A\overline{\omega}_{BC} &= 0.
\end{aligned}$$

Now we discuss properties of Lagrangian submanifolds

2.2. **Lagrangian submanifolds in Kähler manifolds.**
Assume that $L$ is an $n$-dimensional manifold smoothly immersed into a Kähler manifold $(\overline{M}, \overline{J}, \overline{g})$ by a smooth map

$$F : L \to (\overline{M}, \overline{J}, \overline{g}).$$

If $(x^i)_{i=1,\dots,n}$ are local coordinates for $L$, then we set

$$\begin{aligned}
F_i &:= \frac{\partial F}{\partial x^i},\\
\nu_i &:= \overline{J}F_i,\\
F_{ij} &:= \frac{\partial^2 F}{\partial x^i \partial x^j}.
\end{aligned}$$

By definition we have:

$$L_0 := F(L) \text{ is Lagrangian} :\Leftrightarrow \overline{\omega}^* := F^*\omega = 0.$$

Note, that this implies that $\nu_i$ is a normal vector for any $i = 1, \dots, n$. In the sequel we will often rise and lower indices. This will **always** be done w.r.t. the induced metric tensors $g^{ij}, g_{ij} = \bar{g}(F_i, F_j)$, e.g. $h^k{}_{ij} = g^{kl}h_{lij}$. There is, however, one exception, namely if $\sigma_{ij}$ is another metric tensor, then $\sigma^{ij} = (\sigma_{ij})^{-1} \neq g^{ik}g^{jl}\sigma_{kl}$.

The **second fundamental form** on $L$ can then be defined as

$$h_{ijk} := -\overline{g}(\nu_i, \overline{\nabla}_{F_j}F_k)$$

and the **mean curvature** 1-**form** $H_i dx^i$ is given by

$$H_i := g^{kl} h_{ikl}.$$

We also introduce

$$\begin{aligned} A_{ijkl} &:= h_{ijn} h^n{}_{kl}, \\ a_{kl} &:= A_i{}^i{}_{kl} = H^n h_{nkl}, \\ b_{kl} &:= A_k{}^i{}_{li} = h_k{}^{ij} h_{ijl}. \end{aligned}$$

The Christoffel symbols for the induced Levi-Civita connection on $L$ will be written without a bar, i.e. in the form $\Gamma^i_{jk}$. To distinguish objects on the ambient space from the corresponding induced objects on $L$ we will often use a bar for the ambient objects, e.g. $\overline{R}_{ABCD}$ and $R_{ijkl}$ are the curvature tensors on $\overline{M}$ resp. $L$. We also define

$$\overline{R}_{ijkl} := \overline{R}(F_i, F_j, F_k, F_l) = \overline{R}_{ABCD} \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j} \frac{\partial F^C}{\partial x^k} \frac{\partial F^D}{\partial x^l},$$

$$\overline{R}_{ijk\underline{l}} := \overline{R}(F_i, F_j, F_k, \bar{J} F_l).$$

We also write

$$\overline{\nabla}_i \overline{R}_{jklm} = \overline{\nabla}\,\overline{R}(F_i, F_j, F_k, F_l, F_m)$$

and

$$\bar{R}_{ij} = \overline{R}_{AB} \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j}.$$

Note that this is different from $g^{kl} \overline{R}_{ikjl}$. Then we obtain the following equations:

**Proposition 2.2.**

| | | | |
|---|---|---|---|
| **Full symmetry** | : | $h_{ijk} = h_{jik} = h_{jki}$, | (2.4) |
| **Gauss formula** | : | $-h_{jk}{}^n \nu_n = F_{kj} - \Gamma^n_{jk} F_n + \overline{\Gamma}_{AB} F^A_j F^B_k$ | (2.5) |
| $h_{kij}$ | $=$ | $\overline{\omega}(F_{ij} + \overline{\Gamma}_{AB} F^A_i F^B_j, F_k)$, | (2.6) |
| **Gauss equation** | : | $R_{ijkl} = \overline{R}_{ijkl} + A_{ikjl} - A_{iljk}$, | (2.7) |
| **Codazzi equation** | : | $\nabla_i h_{jkl} - \nabla_j h_{ikl} = \overline{R}_{ijk\underline{l}}$, | (2.8) |
| **traced Codazzi eq.** | : | $\nabla_k H_l - \nabla_l H_k = \overline{R}_{\underline{k}l}$, | (2.9) |

*where* $\nabla$ *is the Levi-Civita connection w.r.t.* $g$.

Applying the Codazzi equation and the rule for interchanging derivatives one also obtains the following Simons type identity that will become important in the sequel.

**Lemma 2.3.**

$$
\begin{aligned}
\nabla_i\nabla_j H_k &= \Delta h_{ijk}\\
&- a_i{}^s h_{sjk} + b_i{}^s h_{sjk} + b_j{}^s h_{ski} + b_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n\\
&+ 2h_{ins}\overline{R}^{n\ s}_{\ j\ k} + 2h_{jns}\overline{R}^{n\ s}_{\ k\ i} + 2h_{kns}\overline{R}^{n\ s}_{\ i\ j}\\
&- h_{ij}{}^s\overline{R}_k{}^n{}_{sn} - h_{jk}{}^s\overline{R}_i{}^n{}_{sn} - h_{ki}{}^s\overline{R}_j{}^n{}_{sn}\\
&- H^s\overline{R}_{i\underline{s}j\underline{k}} + h_{ij}{}^s\overline{R}_{sk}\\
&+ \overline{\nabla}_i\overline{R}_j{}^n{}_{n\underline{k}} + \overline{\nabla}_n\overline{R}_i{}^n{}_{j\underline{k}}.
\end{aligned} \tag{2.10}
$$

**Proof:** If we use the Codazzi equation (2.8) and the rule for interchanging derivatives, then we obtain

$$
\begin{aligned}
\nabla_i\nabla_j H_k &= \nabla_i\nabla_j h^n{}_{nk}\\
&= \nabla_i\nabla_n h_j{}^n{}_k + \nabla_i\overline{R}_j{}^n{}_{n\underline{k}}\\
&= \nabla_n\nabla_i h_j{}^n{}_k + R^s{}_j{}^n{}_i h_{snk} + R^s{}_n{}^n{}_i h_{jsk}\\
&+ R^s{}_k{}^n{}_i h_{jns} + \nabla_i\overline{R}_j{}^n{}_{n\underline{k}}\\
&= \Delta h_{ijk} + R^s{}_j{}^n{}_i h_{snk} + R^s{}_n{}^n{}_i h_{jsk} + R^s{}_k{}^n{}_i h_{jns}\\
&+ \nabla_i\overline{R}_j{}^n{}_{n\underline{k}} + \nabla_n\overline{R}_i{}^n{}_{j\underline{k}}.
\end{aligned}
$$

Gauss' equation (2.7) gives

$$
R^s{}_i{}^n{}_j h_{snk} = \overline{R}^s{}_i{}^n{}_j h_{snk} + b_k{}^s h_{sij} - h_{in}{}^m h_{jm}{}^s h_{ks}{}^n.
$$

This yields

$$
\begin{aligned}
\nabla_i\nabla_j H_k &= \Delta h_{ijk}\\
&- a_i{}^s h_{sjk} + b_i{}^s h_{sjk} + b_j{}^s h_{ski} + b_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n\\
&- \overline{R}_{in}{}^{sn} h_{sjk} + \overline{R}^s{}_i{}^n{}_j h_{snk} + \overline{R}^s{}_k{}^n{}_i h_{snj}\\
&+ \nabla_i\overline{R}_j{}^n{}_{n\underline{k}} + \nabla_n\overline{R}_i{}^n{}_{j\underline{k}}.
\end{aligned}
$$

Then equations (2.5) implies

$$
\begin{aligned}
\nabla_i\overline{R}_{jmn\underline{k}} &= \overline{\nabla}_i\overline{R}_{jmn\underline{k}} + h_{ij}{}^s\overline{R}_{m\underline{s}n\underline{k}} - h_{im}{}^s\overline{R}_{j\underline{s}n\underline{k}}\\
&+ h_{ik}{}^s\overline{R}_{jmns} - h_{in}{}^s\overline{R}_{jmsk}.
\end{aligned}
$$

As a result

$$\begin{aligned}
\nabla_i\nabla_j H_k &= \Delta h_{ijk}\\
&- a_i{}^s h_{sjk} + b_i{}^s h_{sjk} + b_j{}^s h_{ski} + b_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n\\
&- \overline{R}_{in}{}^{sn} h_{sjk} + \overline{R}^s{}_i{}^n{}_j h_{snk} + \overline{R}^s{}_k{}^n{}_i h_{snj}\\
&+ h_{ij}{}^s \overline{R}^n{}_{\underline{s}n\underline{k}} - h_{in}{}^s \overline{R}_{j\underline{s}}{}^n{}_{\underline{k}} + h_{ik}{}^s \overline{R}_j{}^n{}_{ns} - h_{in}{}^s \overline{R}_j{}^n{}_{sk}\\
&+ h_{ni}{}^s \overline{R}^n{}_{\underline{s}j\underline{k}} - H^s \overline{R}_{i\underline{s}j\underline{k}} + h_{nk}{}^s \overline{R}_i{}^n{}_{js} - h_{nj}{}^s \overline{R}_i{}^n{}_{sk}\\
&+ \overline{\nabla}_i \overline{R}_j{}^n{}_{n\underline{k}} + \overline{\nabla}_n \overline{R}_i{}^n{}_{j\underline{k}}.
\end{aligned}$$

The first Bianchi identity (2.1) and the Kähler identity (2.2) give

$$-h_{in}{}^s \overline{R}_{j\underline{s}}{}^n{}_{\underline{k}} + h_{ni}{}^s \overline{R}^n{}_{\underline{s}j\underline{k}} = h_{ins} \overline{R}^n{}_j{}^s{}_k$$

and

$$h_{ij}{}^s \overline{R}^n{}_{\underline{s}n\underline{k}} = -h_{ij}{}^s \overline{R}_k{}^n{}_{sn} + h_{ij}{}^s \overline{R}^n{}_{\underline{n}k\underline{s}}.$$

Equation (2.3) and the Kähler identity imply that

$$\overline{R}^n{}_{\underline{n}k\underline{s}} = \overline{R}_{\underline{s}k\underline{n}}{}^n = \frac{1}{2}\overline{R}_{\underline{s}k\underline{\gamma}}{}^\gamma = \overline{R}_{sk}.$$

Inserting the last three equations in the formula for $\nabla_i\nabla_j H_k$ we obtain the result. ∎

2.3. **Lagrangian submanifolds induced by symplectomorphisms.** Let $(L, J, \sigma)$ and $(M, K, \eta)$ be two $n$-dimensional compact Kähler-Einstein manifolds with the same scalar curvature. Local coordinates for $L$ will be denoted by $x^i, i = 1, \dots, n$ and local coordinates for $M$ with $y^\alpha, \alpha = 1, \dots, n$. Moreover $C^\alpha_{\beta\gamma}$ are the Christoffel symbols for $\eta_{\alpha\beta}$ and $\Lambda^i_{jk}$ those for the metric $\sigma_{ij}$. In the following doubled small latin and greek indices are summed from 1 to $n$. We define a new Kähler manifold $(\overline{M}, \overline{J}, \overline{g})$ by

$$\begin{aligned}
\overline{M} &:= L \times M,\\
\overline{J} &:= (J, -K),\\
\overline{g} &:= \sigma \times \eta.
\end{aligned}$$

As before, coordinates for $\overline{M}$ are denoted by $z^A, A = 1, \dots, 2n$. In particular if $x^i, y^\alpha$ are local coordinates around $p \in L$ resp. $q \in M$, then

$$(z^1, \dots, z^{2n}) := (x^1, \dots, x^n, y^1, \dots, y^n)$$

define local coordinates around $(p, q) \in \overline{M}$. The Kähler forms $\omega$, $\widetilde{\omega}$, $\overline{\omega}$ on $L$, $M$ resp. $\overline{M}$ are then given by

$$\omega = \omega_{ij} dx^i \otimes dx^j = J_i{}^k \sigma_{kj} dx^i \otimes dx^j, \tag{2.11}$$

$$\widetilde{\omega} = \widetilde{\omega}_{\alpha\beta} dy^\alpha \otimes dy^\beta = K_\alpha{}^\gamma \eta_{\gamma\beta} dy^\alpha \otimes dy^\beta, \tag{2.12}$$

$$\overline{\omega} = \overline{\omega}_{AB} dz^A \otimes dz^B = \omega_{ij} dx^i \otimes dx^j - \widetilde{\omega}_{\alpha\beta} dy^\alpha \otimes dy^\beta. \tag{2.13}$$

An easy calculations shows

$$\begin{aligned}
\overline{\Gamma}^A_{i(n+\alpha)} &= 0,\\
\overline{\Gamma}^{(n+\alpha)}_{ij} &= 0,\\
\overline{\Gamma}^i_{(n+\alpha)(n+\beta)} &= 0,\\
\overline{\Gamma}^i_{jk} &= \Lambda^i_{jk},\\
\overline{\Gamma}^{(n+\alpha)}_{(n+\beta)(n+\gamma)} &= C^\alpha_{\beta\gamma}.
\end{aligned}$$

Now let $f : L \to M$ be a smooth map. $f$ is called symplectic, if

$$\omega = \widetilde{\omega}^* := f^*\widetilde{\omega}.$$

One observes that this is true if and only if the graph of $f$ in $(\overline{M}, \overline{\omega})$ is Lagrangian, i.e. if and only if the map

$$\begin{aligned}
F &: L \to \overline{M},\\
F(p) &:= (p, f(p))
\end{aligned}$$

defines a Lagrangian immersion of $L$ into $\overline{M}$. Let $f$ be a symplectic map and let $g := F^*\overline{g}$ be the pull back of $\overline{g}$ to $L$. In local coordinates $x^i, y^\alpha$ this yields

$$\begin{aligned}
F(x) &= (x, f(x)),\\
g_{ij} &= \sigma_{ij} + \eta_{\alpha\beta} f^\alpha_i f^\beta_j = \sigma_{ij} + \eta^*_{ij},
\end{aligned}$$

where $\eta^*_{ij} := \eta_{\alpha\beta} f^\alpha_i f^\beta_j$. The curvature tensors w.r.t. $\sigma$ and $\eta$ will be denoted by $S_{ijkl}, T_{\alpha\beta\gamma\delta}$. It is convenient to define the tensor

$$d^k{}_{ij} := J_n{}^k h^n{}_{ij}. \tag{2.14}$$

For $1 \le A \le n$ we observe that $F^A_i = \frac{\partial x^A}{\partial x^i} = \delta_i{}^A$ and $F^A_{ij} = 0$. Hence, when we use (2.5) for these components, then we obtain the following important information:

$$\Gamma^k_{ij} - \Lambda^k_{ij} = d^k{}_{ij}. \tag{2.15}$$

Note that this is a tensor equation. Equation (2.15) allows us to compare covariant derivatives w.r.t. the induced metric $g_{ij}$ and w.r.t. the fixed background metric $\sigma_{ij}$. Covariant derivatives w.r.t. $\sigma_{ij}$ will be denoted by a capital $D$, e.g.

$$D_k g_{ij} = \frac{\partial}{\partial x^k} g_{ij} - \Lambda^l_{ki} g_{lj} - \Lambda^l_{kj} g_{il}$$

and those w.r.t. $g_{ij}$ by $\nabla$. Then $\nabla_k g_{ij} = 0$ and (2.15) imply

$$D_k g_{ij} = d_{ijk} + d_{jik} \tag{2.16}$$

and in the same way $D_k J_i{}^l = 0$ implies

$$\nabla_k J_i{}^l = h^l{}_{ki} + d^l{}_{kn} J_i{}^n. \tag{2.17}$$

We define another 1-form by

$$\lambda_i := g_{ik} J_l{}^k H^l = d_i{}^k{}_k. \tag{2.18}$$

**Lemma 2.4.** *Let $f : L \to M$ be a symplectic map and let $F$ be defined as above. Then the following relations hold:*

$$\begin{aligned} \sigma^{in} &= g^{in} + g^{kl} J_k{}^i J_l{}^n, &(2.19)\\ g^{in} J_n{}^j - g^{jn} J_n{}^i &= \sigma^{in} J_n{}^j, &(2.20)\\ g^{kl}\sigma_{kl} &= \frac{n}{2}, &(2.21)\\ \sigma^{in} g_{in} &= n + |J_i{}^j|^2. &(2.22) \end{aligned}$$

**Proof:** Let $p \in F(L) \subset \overline{M}$ be an arbitrary point and let $V, W \in T_p\overline{M}$ be arbitrary vectors. Since $F(L)$ is Lagrangian one can write these vectors as

$$\begin{aligned} V &= g^{kl}(\overline{g}(V, F_k)F_l + \overline{g}(V, \nu_k)\nu_l),\\ W &= g^{kl}(\overline{g}(W, F_k)F_l + \overline{g}(W, \nu_k)\nu_l). \end{aligned}$$

Now let $V = \frac{\partial}{\partial x^i}, W = \frac{\partial}{\partial x^j}$. We compute

$$\begin{aligned} \sigma_{ij} &= \overline{g}(V, W)\\ &= \overline{g}(g^{kl}\sigma_{ki}F_l + g^{kl}\omega_{ki}\nu_l, g^{mn}\sigma_{mj}F_n + g^{mn}\omega_{mj}\nu_n)\\ &= g^{kl}\sigma_{ki}g^{mn}\sigma_{mj}g_{ln} + g^{kl}\omega_{ki}g^{mn}\omega_{mj}g_{ln}\\ &= g^{kl}\sigma_{ki}\sigma_{lj} + g^{kl}\omega_{ki}\omega_{lj} \end{aligned}$$

and then also

$$\sigma^{in} = g^{in} + g^{kl} J_k{}^i J_l{}^n.$$

Multiplying with $J_n{}^j$ gives (2.20). Then with the compatibility condition

$$n = \sigma^{in}\sigma_{in} = g^{in}\sigma_{in} + g^{kl} J_k{}^i J_l{}^n \sigma_{in} = 2g^{in}\sigma_{in}$$

which is (2.21). If we multiply (2.19) with $g_{in}$, then we obtain (2.22). ∎

**Lemma 2.5.** *The following relation holds*

$$\nabla^k d_{ijk} - \nabla_j \lambda_i = g_{is} J_t{}^s \overline{R}^k{}_j{}^t{}_{\underline{k}} + b_{ij} - a_{ij} + d_i^{nk} d_{nkj} - \lambda^n d_{ijn}. \tag{2.23}$$

**Proof:**

$$\begin{aligned} \nabla^k d_{ijk} - \nabla_j \lambda_i &= \nabla^k d_{ijk} - \nabla_j d_i{}^k{}_k\\ &= \nabla^k (g_{is} J_t{}^s h^t{}_{jk}) - \nabla_j (g_{is} J_t{}^s h^{tk}{}_k)\\ &= g_{is} J_t{}^s (\nabla^k h^t{}_{jk} - \nabla_j h^{tk}{}_k)\\ &+ g_{is} h^t{}_{jk}(h^{sk}{}_t + d^{sk}{}_n J_t{}^n) - g_{is} h^{tk}{}_k (h^s{}_{jt} + d^s{}_{jn} J_t{}^n)\\ &\overset{(2.8)}{=} g_{is} J_t{}^s \overline{R}^k{}_j{}^t{}_{\underline{k}} + b_{ij} - a_{ij} + d_i^{nk} d_{nkj} - \lambda^n d_{ijn}. \end{aligned}$$

∎

## 3. Evolution equations

Assume the general situation that a family of Lagrangian submanifolds $L$ in a Kähler-Einstein manifold $\overline{M}$ with constant scalar curvature $\overline{R}$ moves under the evolution equation

$$\frac{\partial}{\partial t}F = -H^i\nu_i + \widetilde{\lambda}^k F_k, \tag{3.1}$$

where $F : L \to \overline{M}$ and $\widetilde{\lambda}_k$ is a smooth 1-form on $L$, depending on $t$. This is the Lagrangian mean curvature flow with an added tangential motion that does not effect the geometry of $L$. We have

$$\begin{aligned}
\frac{\partial}{\partial t}F_i &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial t}F \\
&= \frac{\partial}{\partial x^i}(-H^l\nu_l + \widetilde{\lambda}^l F_l) \\
&= -H^l{}_{,i}\nu_l - H^l\nu_{l,i} + \widetilde{\lambda}^l{}_{,i}F_l + \widetilde{\lambda}^l F_{l,i} \\
&= -H^l{}_{,i}\nu_l - H^l(h^k{}_{li}F_k + \Gamma^k_{li}\nu_k - \overline{\Gamma}_{AB}F_i^A\nu_l^B) \\
&+ \widetilde{\lambda}^l{}_{,i}F_l + \widetilde{\lambda}^l(-h^k{}_{li}\nu_k + \Gamma^k_{li}F_k - \overline{\Gamma}_{AB}F_i^A F_l^B) \\
&= -(\nabla_i H^k + \widetilde{\lambda}^l h_{li}{}^k)\nu_k + H^l\overline{\Gamma}_{AB}F_i^A\nu_l^B \\
&+ (\nabla_i\widetilde{\lambda}^k - a_i{}^k)F_k - \widetilde{\lambda}^l\overline{\Gamma}_{AB}F_i^A F_l^B.
\end{aligned} \tag{3.2}$$

From this we conclude

$$\begin{aligned}
\frac{\partial}{\partial t}F_{ij} &= -(\nabla_i H^k + \widetilde{\lambda}^l h_{li}{}^k)_{,j}\nu_k - (\nabla_i H^k + \widetilde{\lambda}^l h_{li}{}^k)\nu_{k,j} \\
&+ (H^l F_i^A\nu_l^B)_{,j}\overline{\Gamma}_{AB} + (H^l F_i^A\nu_l^B)\overline{\Gamma}_{AB,C}F_j^C \\
&+ (\nabla_i\widetilde{\lambda}^k - a_i{}^k)_{,j}F_k + (\nabla_i\widetilde{\lambda}^k - a_i{}^k)F_{kj} \\
&- (\widetilde{\lambda}^l F_i^A F_l^B)_{,j}\overline{\Gamma}_{AB} - (\widetilde{\lambda}^l F_i^A F_l^B)\overline{\Gamma}_{AB,C}F_j^C.
\end{aligned} \tag{3.3}$$

In the sequel it will be convenient to assume that at a fixed point $(x, F(x)) \in L \times \overline{M}$ we have chosen a double normal coordinate system, i.e. normal coordinates $x^i$ w.r.t. $g_{ij}$ around $x \in L$ and normal coordinates $z^A$ w.r.t. $\overline{g}_{AB}$ around $F(x) \in \overline{M}$. In such a coordinate system all partial derivatives of the metric tensors $g_{ij}$ and $\overline{g}_{AB}$ vanish at $x$ resp. $F(x)$. In particular all Christoffel symbols $\Gamma^i_{jk}, \overline{\Gamma}^A_{BC}$ vanish and the covariant derivative of any tensor on $L$ coincides with its partial derivative at the point $x$. Then (2.5), (3.2) and (3.3) imply that in double normal coordinates around $x$ the following identities hold at $x$

$$\begin{aligned}
F_{ij} &= -h^k{}_{ij}\nu_k,\\
\nu_{i,j} &= h^k{}_{ij}F_k,\\
\frac{\partial}{\partial t}F_i &= -(\nabla_i H^k + \widetilde{\lambda}^l h_{li}{}^k)\nu_k + (\nabla_i\widetilde{\lambda}^k - a_i{}^k)F_k\\
\frac{\partial}{\partial t}F_{ij} &= -\big(\nabla_j\nabla_i H^k + \nabla_j(\widetilde{\lambda}^l h_{li}{}^k) + (\nabla_i\widetilde{\lambda}^l - a_i{}^l)h_{lj}{}^k\big)\nu_k\\
&+ \big(\nabla_j\nabla_i\widetilde{\lambda}^l - \nabla_j a_i{}^k - (\nabla_i H^l + \widetilde{\lambda}^n h_{ni}{}^l)h_{lj}{}^k\big)F_k\\
&- \overline{\Gamma}_{AB,C}F_i^A F_j^C\frac{\partial}{\partial t}F^B.
\end{aligned}$$

**Lemma 3.1.** *The metric $g_{ij}$ satisfies the evolution equation*

$$\frac{\partial}{\partial t}g_{ij} = -2a_{ij} + \nabla_i\widetilde{\lambda}_j + \nabla_j\widetilde{\lambda}_i. \tag{3.4}$$

**Proof:** We use double normal coordinates at $x$ to compute

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= \overline{g}(\frac{\partial}{\partial t}F_i, F_j) + \overline{g}(F_i, \frac{\partial}{\partial t}F_j)\\
&= -2a_{ij} + \nabla_i\widetilde{\lambda}_j + \nabla_j\widetilde{\lambda}_i.
\end{aligned}$$

Since this equation is coordinate independent it holds on all of $L$. ■

In the same way we proceed with $h_{ijk}$.

**Lemma 3.2.** *The second fundamental form satisfies the evolution equations*

$$\begin{aligned}
\frac{\partial}{\partial t}h_{kij} &= \nabla_j\nabla_i H_k - a_i{}^l h_{ljk} - a_{kl}h^l{}_{ij} + H^l\overline{R}_{\underline{l}j\underline{k}i}\\
&+ \nabla_k\widetilde{\lambda}^l h_{lij} + \nabla_i\widetilde{\lambda}^l h_{ljk} + \nabla_j\widetilde{\lambda}^l h_{lki} + \widetilde{\lambda}^l\nabla_l h_{kij}, \qquad (3.5)\\
\frac{\partial}{\partial t}h_{ijk} &= \Delta h_{ijk}\\
&- R_i{}^s h_{sjk} - R_j{}^s h_{ski} - R_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n\\
&+ 2h_{ins}\overline{R}^n{}_j{}^s{}_k + 2h_{jns}\overline{R}^n{}_k{}^s{}_i + 2h_{kns}\overline{R}^n{}_i{}^s{}_j + h_{ij}{}^s\overline{R}_{sk}\\
&+ \overline{\nabla}_i\overline{R}_j{}^n{}_{n\underline{k}} + \overline{\nabla}_n\overline{R}_i{}^n{}_{j\underline{k}}\\
&+ \nabla_k\widetilde{\lambda}^l h_{lji} + \nabla_j\widetilde{\lambda}^l h_{lik} + \nabla_i\widetilde{\lambda}^l h_{lkj} + \widetilde{\lambda}^l\nabla_l h_{kji}. \qquad (3.6)
\end{aligned}$$

**Proof:** We can use equation (2.6) to compute

$$\begin{aligned}
\frac{\partial}{\partial t}h_{kij} &= \overline{\omega}(\frac{\partial}{\partial t}F_{ij} + \overline{\Gamma}_{AB,C}\frac{\partial}{\partial t}F^C F_i^A F_j^B, F_k) + \overline{\omega}(-h^l{}_{ij}\nu_l, \frac{\partial}{\partial t}F_k)\\
&= -\big(\nabla_j\nabla_i H^m + \nabla_j(\widetilde{\lambda}^l h_{li}{}^m) + (\nabla_i\widetilde{\lambda}^l - a_i{}^l)h_{lj}{}^m\big)\overline{\omega}(\nu_m, F_k)\\
&+ \overline{\omega}((\overline{\Gamma}_{AB,C} - \overline{\Gamma}_{AC,B})\frac{\partial}{\partial t}F^C F_i^A F_j^B, F_k)\\
&- h^l{}_{ij}(\nabla_k\widetilde{\lambda}^m - a_k{}^m)\overline{\omega}(\nu_l, F_m)\\
&= \nabla_j\nabla_i H_k + \nabla_j(\widetilde{\lambda}^l h_{lik}) + (\nabla_i\widetilde{\lambda}^l - a_i{}^l)h_{ljk} + h^l{}_{ij}(\nabla_k\widetilde{\lambda}_l - a_{kl})\\
&+ \overline{\omega}((\overline{\Gamma}_{AB,C} - \overline{\Gamma}_{AC,B})\frac{\partial}{\partial t}F^C F_i^A F_j^B, F_k)
\end{aligned}$$

In normal coordinates one has

$$\overline{\Gamma}_{AB,C} - \overline{\Gamma}_{AC,B} = -\overline{R}^D{}_{ABC}\frac{\partial}{\partial z^D}.$$

This implies

$$\begin{aligned}
\frac{\partial}{\partial t}h_{kij} &= \nabla_j\nabla_i H_k + \nabla_j(\widetilde{\lambda}^l h_{lik}) + (\nabla_i\widetilde{\lambda}^l - a_i{}^l)h_{ljk} + h^l{}_{ij}(\nabla_k\widetilde{\lambda}_l - a_{kl})\\
&+ \overline{R}_{\underline{k}ijC}\frac{\partial}{\partial t}F^C.\\
&= \nabla_j\nabla_i H_k - a_i{}^l h_{ljk} - a_{kl}h^l{}_{ij}\\
&+ \nabla_k\widetilde{\lambda}^l h_{lij} + \nabla_i\widetilde{\lambda}^l h_{ljk} + \nabla_j\widetilde{\lambda}^l h_{lki}\\
&+ \widetilde{\lambda}^l\nabla_j h_{lik} - H^l\overline{R}_{\underline{k}ij\underline{l}} + \widetilde{\lambda}^l\overline{R}_{\underline{k}ijl}\\
&= \nabla_j\nabla_i H_k - a_i{}^l h_{ljk} - a_{kl}h^l{}_{ij} + H^l\overline{R}_{\underline{l}j\underline{k}i}\\
&+ \nabla_k\widetilde{\lambda}^l h_{lij} + \nabla_i\widetilde{\lambda}^l h_{ljk} + \nabla_j\widetilde{\lambda}^l h_{lki} + \widetilde{\lambda}^l\nabla_l h_{kij},
\end{aligned}$$

where we used the Codazzi equation in the last step. This gives (3.5). We apply Lemma 2.3 to (3.5) and obtain

$$\begin{aligned}
\frac{\partial}{\partial t}h_{kji} &= \Delta h_{ijk}\\
&- a_i{}^s h_{sjk} + b_i{}^s h_{sjk} + b_j{}^s h_{ski} + b_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n\\
&+ 2h_{ins}\overline{R}^n{}_j{}^s{}_k + 2h_{jns}\overline{R}^n{}_k{}^s{}_i + 2h_{kns}\overline{R}^n{}_i{}^s{}_j\\
&- h_{ij}{}^s\overline{R}_k{}^n{}_{sn} - h_{jk}{}^s\overline{R}_i{}^n{}_{sn} - h_{ki}{}^s\overline{R}_j{}^n{}_{sn}\\
&- H^s\overline{R}_{i\underline{s}j\underline{k}} + h_{ij}{}^s\overline{R}_{sk}\\
&+ \overline{\nabla}_i\overline{R}_j{}^n{}_{n\underline{k}} + \overline{\nabla}_n\overline{R}_i{}^n{}_{j\underline{k}}\\
&- a_j{}^l h_{lik} - a_{kl}h^l{}_{ji} + H^l\overline{R}_{\underline{l}i\underline{k}j}\\
&+ \nabla_k\widetilde{\lambda}^l h_{lji} + \nabla_j\widetilde{\lambda}^l h_{lik} + \nabla_i\widetilde{\lambda}^l h_{lkj} + \widetilde{\lambda}^l\nabla_l h_{kji}.
\end{aligned}$$

From the Gauß curvature equations we know that

$$-R_{is} = -a_{is} + b_{is} - \overline{R}_i{}^n{}_{sn}$$

Inserting this into the equation above we deduce

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ijk} &= \Delta h_{ijk} \\
&- R_i{}^s h_{sjk} - R_j{}^s h_{ski} - R_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n \\
&+ 2h_{ins}\overline{R}^n{}_j{}^s{}_k + 2h_{jns}\overline{R}^n{}_k{}^s{}_i + 2h_{kns}\overline{R}^n{}_i{}^s{}_j + h_{ij}{}^s\overline{R}_{sk} \\
&+ \overline{\nabla}_i\overline{R}_j{}^n{}_{n\underline{k}} + \overline{\nabla}_n\overline{R}_i{}^n{}_{j\underline{k}} \\
&+ \nabla_k\widetilde{\lambda}^l h_{lji} + \nabla_j\widetilde{\lambda}^l h_{lik} + \nabla_i\widetilde{\lambda}^l h_{lkj} + \widetilde{\lambda}^l\nabla_l h_{kji}.
\end{aligned}$$

■

**Lemma 3.3.** *The mean curvature form satisfies the evolution equation*

$$\frac{\partial}{\partial t}H = d(d^\dagger H + \langle H, \widetilde{\lambda}\rangle) + \frac{\overline{R}}{2n}H, \tag{3.7}$$

*where* $d^\dagger H := \nabla^i H_i$ *and* $\overline{R}$ *is the scalar curvature of the* $2n$*-dimensional manifold* $(\overline{M}, \overline{g})$.

**Proof:** Equations (3.4) and (3.5) imply

$$\begin{aligned}
\frac{\partial}{\partial t}H_j &= \frac{\partial}{\partial t}(g^{ik}h_{kij}) \\
&= (2a^{ik} - 2\nabla^k\widetilde{\lambda}^i)h_{kij} + \nabla_j d^\dagger H - 2a^{ik}h_{kij} + H^l\overline{R}_{\underline{l}j\underline{k}i}g^{ki} \\
&+ 2\nabla^k\widetilde{\lambda}^i h_{kij} + \nabla_j\widetilde{\lambda}^l H_l + \widetilde{\lambda}^l\nabla_j H_l \\
&= \nabla_j(d^\dagger H + \langle H, \widetilde{\lambda}\rangle) + H^l\overline{R}_{\underline{l}j\underline{k}i}g^{ki}.
\end{aligned}$$

Since $g_{ij} = \overline{g}(F_i, F_j) = \overline{g}(\nu_i, \nu_j)$ we deduce from (2.3) that

$$\overline{R}_{\underline{l}j\underline{k}i}g^{ki} = \overline{R}_{lj} = \frac{\overline{R}}{2n}g_{lj}$$

which implies the lemma. ■

As a corollary we obtain

**Lemma 3.4.**

$$\begin{aligned}
\frac{\partial}{\partial t}|H|^2 &= \Delta|H|^2 + \langle\widetilde{\lambda}, \nabla|H|^2\rangle - 2|\nabla H|^2 \\
&+ 2b^{ij}H_iH_j - 2\overline{R}^{imj}{}_m H_iH_j + \frac{\overline{R}}{n}|H|^2.
\end{aligned}$$

Similarly as in [5] we obtain

**Lemma 3.5.** *If the initial mean curvature form $H$ is exact, then there exists a unique smooth angle function $\alpha$ such that*

$$\begin{aligned} d\alpha &= H, \\ osc(\alpha) &= 2max(\alpha), \;\; at\ t=0 \\ \frac{\partial}{\partial t}\alpha &= \Delta\alpha + \langle\widetilde{\lambda}, \nabla\alpha\rangle + \frac{\overline{R}}{2n}\alpha. \end{aligned} \tag{3.8}$$

Next we want to compute the evolution for $|A|^2 = h^{ijk}h_{ijk}$.

**Lemma 3.6.** $|A|^2$ *satisfies*

$$\begin{aligned} \frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2 + \langle\widetilde{\lambda}, \nabla|A|^2\rangle - 2|\nabla A|^2 \\ &+ 2|b_{ij}|^2 + 2|A_{injs} - A_{ijns}|^2 \\ &+ 12\overline{R}^{njsk}A_{jkns} + 2\overline{R}^{ij}b_{ij} - 6\overline{R}^{ni\ \ j}_{\ \ \ n}b_{ij} \\ &+ 2h^{ijk}(\overline{\nabla}_i\overline{R}_{j\ n\underline{k}}^{\ n} + \overline{\nabla}_n\overline{R}_{i\ j\underline{k}}^{\ n}). \end{aligned} \tag{3.9}$$

**Proof:**

$$\begin{aligned} \frac{\partial}{\partial t}|A|^2 &= 2h^{ijk}\frac{\partial}{\partial t}h_{ijk} + 3\frac{\partial}{\partial t}g^{ij}b_{ij} \\ &\overset{(3.6)}{=} \Delta|A|^2 - 2|\nabla A|^2 - 6R^{ij}b_{ij} - 4A_{injs}A^{ijns} \\ &+ 12\overline{R}^{njsk}A_{jkns} + 2\overline{R}^{ij}b_{ij} \\ &+ 2h^{ijk}(\overline{\nabla}_i\overline{R}_{j\ n\underline{k}}^{\ n} + \overline{\nabla}_n\overline{R}_{i\ j\underline{k}}^{\ n}) \\ &+ 6b^{ij}\nabla_i\widetilde{\lambda}_j + \widetilde{\lambda}^l\nabla_l|A|^2 + 6(a^{ij} - \nabla^i\widetilde{\lambda}^j)b_{ij} \\ &= \Delta|A|^2 - 2|\nabla A|^2 + \langle\widetilde{\lambda}, \nabla|A|^2\rangle \\ &+ 2|b_{ij}|^2 + 2|A_{injs} - A_{ijns}|^2 \\ &+ 12\overline{R}^{njsk}A_{jkns} + 2\overline{R}^{ij}b_{ij} - 6\overline{R}^{ni\ \ j}_{\ \ \ n}b_{ij} \\ &+ 2h^{ijk}(\overline{\nabla}_i\overline{R}_{j\ n\underline{k}}^{\ n} + \overline{\nabla}_n\overline{R}_{i\ j\underline{k}}^{\ n}). \end{aligned}$$

■

3.1. **The condition $\overline{\kappa} > 0$.**
Let us assume that $J, K$ are two complex structures on a 4-dimensional Kähler-Einstein manifold $(\overline{M}, \overline{g})$ such that

$$JK = KJ.$$

It follows that

$$JKJK = JKKJ = \mathrm{Id}$$

and

$$JK(JV) = J^2KV = -KV = J(JKV)$$

which means that the eigenvalues of $JK$ are either 1 or $-1$ and that the eigenspaces of $JK$ are holomorphic w.r.t. $J$ and $K$. Moreover $K$ must be compatible with $\overline{g}$ since

$$\overline{g}(KV, KW) = \overline{g}(JKV, JKW) = \overline{g}(V, W).$$

In particular if $J \neq K$ and $J \neq -K$, then the eigenvalues are 1 and $-1$ with multiplicity 2. Using complex coordinates for $J$, it easily follows that $\overline{M}$ locally splits into a product manifold $L \times M$. Now let $L$ be an oriented Lagrangian surface w.r.t. $J$. The Kähler form $\overline{\kappa}$ w.r.t. $K$ is then given by $\overline{\kappa}(V, W) = \overline{g}(KV, W)$. The restriction of a closed 2-form to an oriented surface $L$ with volume form $\mu$ defines a function $r$ by $\kappa := \overline{\kappa}_{|TL} = r\mu$. We assume that $L$ evolves according to equation (3.1) with $\widetilde{\lambda} = 0$ and want to compute the evolution equation for $r$. First we use $\overline{\nabla}\overline{\kappa} = 0$ to observe

$$\nabla_i \kappa_{jk} \;=\; -h_{ij}{}^{n}\tau_{nk} + h_{ik}{}^{n}\tau_{nj}$$

with $\tau(V, W) := \overline{\kappa}(JV, W)$. We also observe that

$$\overline{\kappa}(JV, JW) = \overline{g}(KJV, JW) = \overline{g}(JKV, JW) = \overline{g}(KV, W) = \overline{\kappa}(V, W)$$

from which it follows that

$$\nabla_l \tau_{nk} \;=\; h_{ln}{}^{m}\kappa_{mk} - h_{lk}{}^{m}\kappa_{nm}$$

and then also

$$\begin{aligned}
\Delta\kappa_{jk} \;&=\; -\nabla^i h_{ij}{}^{n}\tau_{nk} + \nabla^i h_{ik}{}^{n}\tau_{nj} - h^{l}{}_{j}{}^{n}(h_{ln}{}^{m}\kappa_{mk} - h_{lk}{}^{m}\kappa_{nm})\\
&+\; h^{l}{}_{k}{}^{n}(h_{ln}{}^{m}\kappa_{mj} - h_{lj}{}^{m}\kappa_{nm})\\
&=\; -\nabla^i h_{ij}{}^{n}\tau_{nk} + \nabla^i h_{ik}{}^{n}\tau_{nj}\\
&-\; b_j{}^{l}\kappa_{lk} + b_k{}^{l}\kappa_{lj} + (A_j{}^{n}{}_{k}{}^{m} - A_k{}^{n}{}_{j}{}^{m})\kappa_{nm}
\end{aligned}$$

Since the Lagrangian submanifold is 2-dimensional the symmetry properties of $A_j{}^{n}{}_{k}{}^{m} - A_k{}^{n}{}_{j}{}^{m}$ imply

$$\begin{aligned}
(A_j{}^{n}{}_{k}{}^{m} - A_k{}^{n}{}_{j}{}^{m})\kappa_{nm} \;&=\; \frac{1}{2}(|H|^2 - |A|^2)(\delta_j{}^{n}\delta_k{}^{m} - \delta_k{}^{n}\delta_j{}^{m})\kappa_{nm}\\
&=\; (|H|^2 - |A|^2)\kappa_{jk} = (|H|^2 - |A|^2)r\mu_{jk}.
\end{aligned}$$

This gives

$$\Delta\kappa_{jk} = -\nabla^i h_{ij}{}^{n}\tau_{nk} + \nabla^i h_{ik}{}^{n}\tau_{nj} - b_j{}^{l}\kappa_{lk} + b_k{}^{l}\kappa_{lj} + (|H|^2 - |A|^2)r\mu_{jk}. \tag{3.10}$$

On the other hand we conclude from

$$\kappa_{jk} = r\mu_{jk}$$

that

$$\Delta\kappa_{jk} = \Delta r\mu_{jk}. \tag{3.11}$$

For the time derivative we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\kappa_{jk} &= \overline{\kappa}(\nabla_j(-H^n\nu_n), F_k) + \overline{\kappa}(F_j, \nabla_k(-H^n\nu_n)) \\ &= -\nabla_j H^n \tau_{nk} + \nabla_k H^n \tau_{nj} - a_j{}^l \kappa_{lk} + a_k{}^l \kappa_{lj}. \qquad (3.12)\end{aligned}$$

On the other hand

$$\frac{\partial}{\partial t}\kappa_{jk} = (\frac{\partial}{\partial t}r - |H|^2 r)\mu_{jk} \qquad (3.13)$$

since the volume element evolves by $\frac{\partial}{\partial t}\mu = -|H|^2\mu$. Combining equations (3.10)–(3.13) we get

$$\begin{aligned}(\frac{\partial}{\partial t}r - \Delta r - |H|^2 r)\mu_{jk} &= -(\nabla_j H^n - \nabla^l h_{lj}{}^n)\tau_{nk} + (\nabla_k H^n - \nabla^l h_{lk}{}^n)\tau_{nj} \\ &+ (b_j{}^l - a_j{}^l)\kappa_{lk} - (b_k{}^l - a_k{}^l)\kappa_{lj} \\ &+ (|A|^2 - |H|^2) r \mu_{jk}.\end{aligned}$$

Since $b_j{}^l - a_j{}^l = \frac{1}{2}(|A|^2 - |H|^2)\delta_j{}^l$ we obtain with the Codazzi equation

$$\begin{aligned}(\frac{\partial}{\partial t}r - \Delta r - |H|^2 r)\mu_{jk} &= -\overline{R}^{l\ n}_{\ j\ \underline{l}}\tau_{nk} + \overline{R}^{l\ n}_{\ k\ \underline{l}}\tau_{nj} \\ &+ 2(|A|^2 - |H|^2) r \mu_{jk}.\end{aligned}$$

If $\overline{R}_{\alpha\beta} = \frac{\overline{R}}{2n}\overline{g}_{\alpha\beta} = \frac{\overline{R}}{4}\overline{g}_{\alpha\beta}$, then the decomposition of the tangent spaces as above and the property of $\overline{M}$ being Kähler w.r.t. both complex structures gives

$$-\overline{R}^{l\ n}_{\ j\ \underline{l}}\tau_{nk} + \overline{R}^{l\ n}_{\ k\ \underline{l}}\tau_{nj} = \frac{\overline{R}}{4}r(1 - r^2)\mu_{jk}$$

so that we derive

**Lemma 3.7.** *The quantity $r$ evolves according to*

$$\frac{\partial}{\partial t}r = \Delta r + \big(2|A|^2 - |H|^2 + \frac{\overline{R}}{4}(1 - r^2)\big)r.$$

*It will remain positive, if this is true for $t = 0$.*

*Remark* 3.8. In section 5 we will consider symplectic maps $f : (L, \sigma, J_1, \omega) \to (M, \eta, J_2, \widetilde{\omega})$ between two Riemann surfaces of the same constant curvature $S$. Then $F := \mathrm{Id}_L \times f : L \to L \times M$ is Lagrangian w.r.t. $\overline{\omega} = (\omega, -\widetilde{\omega})$ and the two complex structures $K := (J_1, J_2)$ and $J := (J_1, -J_2)$ commute. Whereas the pull back of $\overline{\omega}$ to $L$ vanishes we obtain that $\kappa = F^*(\omega, \widetilde{\omega}) = 2\omega$. The Riemannian volume form $\mu$ for $g := F^*(\sigma, \eta)$ satisfies $\mu = \sqrt{\det g_{ij}}\omega$ and in addition $\kappa = 2\omega = r\mu$ by definition of $r$. So we conclude that the function $p$ which will be introduced in section 5 as the determinant of the induced metric tensor $g$ is related to $r$ by

$$r = \frac{2}{\sqrt{p}}.$$

## 4. Proof of Theorems 1.2 and 1.3

4.1. **Proof of Theorem 1.2.** Let us choose $\widetilde{\lambda} = 0$ in the preceding evolution equations. Then as in [4] we can use the evolution equation for the squared norm of the second fundamental tensor $|A|^2 = |h_{ijk}|^2$ to derive the following inequalities for all $m \geq 0$:

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 &\leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 \\ &+ C(n,m)\{\sum_{i+j+k=m} |\nabla^i A| \cdot |\nabla^j A| \cdot |\nabla^k A| \cdot |\nabla^m A| \\ &+ \widetilde{C}_m \sum_{i\leq m} |\nabla^i A| \cdot |\nabla^m A| + \widetilde{C}_{m+1}|\nabla^m A|\},\end{aligned}$$

where the constants depend only on $n$ and $\overline{M}$. Assume that $m \geq 0$ and that we have already bounds for $|\nabla^l A|^2$ for all $0 \leq l < m$. Then, using Cauchy-Schwarz and the evolution inequalities for the norms, there are constants $a_k, b_k$ such that

$$\frac{\partial}{\partial t}|\nabla^k A|^2 \leq \Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2 + a_k|\nabla^k A|^2 + b_k$$

for any $0 \leq k \leq m$. Define the function $f := c|\nabla^{m-1}A|^2 + |\nabla^m A|^2$. Then $f$ satisfies

$$\frac{\partial}{\partial t} f \leq \Delta f - (2c - a_m)|\nabla^m A|^2 + d_m,$$

where $d_m$ is a constant independent of $c$ and where we used that we already have bounds up to order $m-1$. But then we can choose $2c > a_m$ and see that $f$ must be bounded by the maximum principle. This proves that uniform bounds for $|A|^2$ imply uniform bounds for all $|\nabla^m A|^2$, $m \geq 0$. Hence we proved part (a) of Theorem 1.2. Part (b) is now a direct consequence of (a) since we can continue the solution of the mean curvature flow if $|A|^2$ will stay uniformly bounded on a finite time interval. To prove part (c) we observe that in this case all the assumptions in the Harnack inequality of [1] are satisfied and we may apply this theorem to the rescaled angle function $\widetilde{\alpha} := e^{-\frac{S}{n}t}\alpha$ with $\alpha$ as in Lemma 3.5 and obtain as in that paper

$$\operatorname{osc}(\widetilde{\alpha}) \leq ce^{-\epsilon t}$$

for two positive constants $c, \epsilon$ that do not depend on $t$. This implies

$$\operatorname{osc}(\alpha) \leq ce^{(\frac{S}{n}-\epsilon)t} \tag{4.1}$$

and hence the Lagrangian angle tends to zero if $S \leq 0$ and $t \to \infty$. Since by the maximum principle $\alpha$ is bounded, if $S \leq 0$ we find a constant $\gamma$ such that

$$|\alpha - \gamma| \leq 2ce^{(\frac{S}{n}-\epsilon)t} \tag{4.2}$$

Now, if $(L,g)$ is a compact Riemannian manifold, there exists a constant depending only on $L$ and $g$ such that for any smooth functions $f$ the interpolation inequality

$$|\nabla f|^2 \le c|f|\cdot(|\nabla^2 f|+|\nabla f|)$$

holds. If $g_t$ is a family of uniformly equivalent metrics on $L$, then one can choose $c$ independent of $t$. From this we deduce that

$$|H|^2 \le c|\alpha-\gamma|\cdot(|\nabla H|+|H|)\le \widetilde{c}e^{\frac{S-n\epsilon}{2n}t}$$

since $|\nabla H|^2 \le c(n)|\nabla A|^2 \le \text{const}$ by (a). Consequently the mean curvature vector tends to zero exponentially too. Now we can integrate the evolution equation $\frac{\partial}{\partial t}F = \overrightarrow{H}$ and the exponential decay of $|H|$ shows that for any $\epsilon > 0$ we can find a time $t_0$ such that for all $t \ge t_0$ the immersion $L_t$ will stay in an $\epsilon$-neighbourhood of $L_{t_0}$. Then this proves convergence as in [6].

4.2. **Proof of Theorem 1.3.** To prove the first angle theorem we will use Lemma 3.5. Choose $\alpha$ as in Lemma 3.5 ($\widetilde{\lambda}=0$) and define the function

$$f := \cos(e^{-\frac{S}{n}t}\alpha)$$

Then under the mean curvature flow

$$\frac{\partial}{\partial t}\int_{L_t} f d\mu = \int_{L_t}(-e^{-\frac{S}{n}t}\sin(e^{-\frac{S}{n}t}\alpha)\Delta\alpha - f|H|^2)d\mu.$$

Integration by parts gives

$$\frac{\partial}{\partial t}\int_{L_t} f d\mu = \int_{L_t}(e^{-\frac{2S}{n}t}-1)f|H|^2 d\mu.$$

This quantity is positive if $S \le 0$ because by the maximum principle and the assumption on the oscillation, $\widetilde{\alpha} = e^{-\frac{S}{n}t}\alpha$ stays bounded between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But since $\int_{L_t} d\mu \ge \int_{L_t} f d\mu$ we obtain that $\int_{L_t} d\mu \ge \int_{L_0} f d\mu \ge c > 0$ for some constant $c$. This proves part (b). For (a) we proceed similarly but here we obtain that the Lagrangian angle evolves under Lagrangian motions induced by 1-forms $\theta$ by

$$\frac{\partial}{\partial t}\alpha = \nabla^i\theta_i$$

and the volume form according to

$$\frac{\partial}{\partial t}d\mu = -\langle\theta, H\rangle d\mu.$$

Partial integration as above for $f=\cos(\alpha)$ implies $\frac{\partial}{\partial t}\int f d\mu = 0$ and then we can conclude as before.

## 5. Evolution of symplectic maps

Now assume that $f_t : L \to M$ is a smooth family of symplectic maps that satisfy the evolution equation

$$\frac{\partial}{\partial t} f^\alpha = H^k f^\beta_k K_\beta{}^\alpha + \lambda^k f^\alpha_k. \tag{5.1}$$

Then the maps $F_t := \mathrm{Id}_L \times f_t : L \to \overline{M}$ satisfy the evolution equation

$$\frac{\partial}{\partial t} F = -H^i \nu_i + \lambda^i F_i,$$

i.e. (3.1) with $\widetilde{\lambda} = \lambda$, where $\lambda$ is as in (2.18). Equation (5.1) is a system of quasilinear parabolic equations and if the base manifold $L$ is compact it always admits a smooth solution for a short time. The best way to see the parabolic nature is to compute the normalized Laplacian $\widetilde{\Delta} := g^{kl} D_k D_l$ for the coordinate functions $f^\alpha$. First for any function $f$ one deduces

$$\Delta f = g^{kl} \nabla_k \nabla_l f = g^{kl}(D_k D_l f - d^n{}_{kl} D_n f) = \widetilde{\Delta} f - \langle \lambda, \nabla f \rangle \tag{5.2}$$

and then (2.5) gives

$$\begin{aligned} \Delta f^\alpha &= -H^i \nu^\alpha_i - C^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j = H^i f^\beta_i K_\beta{}^\alpha - C^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j \\ &= \widetilde{\Delta} f^\alpha - \langle \lambda, \nabla f^\alpha \rangle, \end{aligned}$$

therefore

$$\frac{\partial}{\partial t} f^\alpha = \widetilde{\Delta} f^\alpha + C^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j.$$

Up to tangential deformations (3.1) corresponds to the Lagrangian mean curvature flow

$$\frac{d}{dt} F = -H^i \nu_i. \tag{5.3}$$

In fact, if we solve for the integral flows $\phi : L \to L$ of

$$\frac{d}{dt} x^i = -\lambda^i, \tag{5.4}$$

then $\widetilde{F} : L \to \overline{M}$ with $\widetilde{F}(x,t) := F(\phi(x,t),t)$ satisfies

$$\frac{d}{dt}\widetilde{F} = \frac{\partial}{\partial t}\widetilde{F} + \widetilde{F}_i \frac{d}{dt} x^i = -\widetilde{H}^i \widetilde{\nu}_i + \widetilde{\lambda}^i \widetilde{F}_i - \widetilde{\lambda}^i \widetilde{F}_i = -\widetilde{H}^i \widetilde{\nu}_i.$$

Let us define the following crucial function

$$p := \frac{\det(g_{ij})}{\det(\sigma_{ij})} = \det(g_{ik}\sigma^{kj}).$$

Since $g_{ij} \geq \sigma_{ij}$ we conclude that all eigenvalues of $g_{ij}\sigma^{jk}$ are bounded below by 1. Let $\lambda_1, \lambda_2$ be the biggest eigenvalues of $g_{ij}\sigma^{jk}$. Then $g^{ij}\sigma_{ij} \geq \frac{n}{2}(\frac{1}{\lambda_1} + \frac{1}{\lambda_2})$ and with (2.21) $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \leq 1$ and therefore $\lambda_1 \geq \frac{\lambda_2}{\lambda_2 - 1}$. This gives $\lambda_1\lambda_2 \geq \frac{\lambda_2^2}{\lambda_2 - 1} \geq 4$. It follows that $p = \lambda_1 \cdots \lambda_n \geq \lambda_1\lambda_2 \geq 4$.

**Lemma 5.1.** *The function p satisfies the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}p &= \Delta p + \langle\lambda, \nabla p\rangle - \frac{3}{2p}|\nabla p|^2\\ &- 2p\big(|A|^2 + J_n{}^iJ_m{}^j(A^n{}_j{}^m{}_i - A^n{}_i{}^m{}_j) + J_t{}^s\overline{R}^k{}_s{}^t{}_{\underline{k}}\big).\end{aligned} \tag{5.5}$$

**Proof:** Equation (3.4) implies for $g := \det(g_{ij})$

$$\frac{\partial}{\partial t}g = g^{kl}\frac{\partial}{\partial t}g_{kl}\cdot g = 2(d^\dagger\lambda - |H|^2)g$$

and then also

$$\frac{\partial}{\partial t}p = 2(d^\dagger\lambda - |H|^2)p. \tag{5.6}$$

We want to rewrite (3.4) and (5.6) as reaction diffusion equations. To this end we need expressions for the normalized Laplacian of $p$. We compute

$$\begin{aligned}\widetilde{\Delta}g_{ij} &= g^{kl}D_kD_lg_{ij}\\ &= g^{kl}(\nabla_kD_lg_{ij} + d^n{}_{kl}D_ng_{ij} + d^n{}_{ki}D_lg_{nj} + d^n{}_{kj}D_lg_{in})\\ &= \nabla^k(d_{ijk} + d_{jik}) + \lambda^n(d_{ijn} + d_{jin}) + 2d^{nk}{}_id_{nkj}\\ &+ d_i{}^{nk}d_{nkj} + d_j{}^{nk}d_{nki}.\end{aligned}$$

Combining this with Lemma 2.5 we get

$$\begin{aligned}\widetilde{\Delta}g_{ij} &= \nabla_i\lambda_j + \nabla_j\lambda_i - 2a_{ij} + 2b_{ij} + g_{is}J_t{}^s\overline{R}^k{}_j{}^t{}_{\underline{k}} + g_{js}J_t{}^s\overline{R}^k{}_i{}^t{}_{\underline{k}}\\ &+ 2d^{nk}{}_id_{nkj} + 2d_i{}^{nk}d_{nkj} + 2d_j{}^{nk}d_{nki}.\end{aligned}$$

and consequently

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= \widetilde{\Delta}g_{ij} - 2b_{ij} - g_{is}J_t{}^s\overline{R}^k{}_j{}^t{}_{\underline{k}} - g_{js}J_t{}^s\overline{R}^k{}_i{}^t{}_{\underline{k}}\\ &- 2d^{nk}{}_id_{nkj} - 2d_i{}^{nk}d_{nkj} - 2d_j{}^{nk}d_{nki}.\end{aligned} \tag{5.7}$$

We have

$$\begin{aligned}\frac{\partial}{\partial t}p &= g^{ij}\frac{\partial}{\partial t}g_{ij}\cdot p,\\ D_kp &= g^{ij}D_kg_{ij}\cdot p,\\ D_lD_kp &= (-g^{is}g^{jt}D_lg_{st}D_kg_{ij} + g^{ij}D_lD_kg_{ij} + \frac{1}{p^2}D_lpD_kp)\cdot p,\\ \widetilde{\Delta}p &= (-|D_kg_{ij}|^2 + g^{ij}\widetilde{\Delta}g_{ij} + \frac{1}{p^2}|\nabla p|^2)\cdot p.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}p &= \widetilde{\Delta}p + (|D_kg_{ij}|^2 + g^{ij}(\frac{\partial}{\partial t}g_{ij} - \widetilde{\Delta}g_{ij}) - \frac{1}{p^2}|\nabla p|^2)\cdot p\\ &= \widetilde{\Delta}p + (|d_{ijk} + d_{jik}|^2 - \frac{1}{p^2}|\nabla p|^2)\cdot p\\ &- (2|A|^2 + 2J_t{}^s\overline{R}^k{}_s{}^t{}_{\underline{k}} + 2|d_{ijk}|^2 + 4d_{ijk}d^{jki})\cdot p,\end{aligned}$$

i.e.

$$\frac{\partial}{\partial t}p = \widetilde{\Delta}p - \frac{1}{p}|\nabla p|^2 - 2p(|A|^2 + J_t{}^s\overline{R}^k{}_s{}^t{}_{\underline{k}} + d_{ijk}d^{jki}).$$

Next we observe that

$$\begin{aligned} d_{ijk}d^{jki} &= J_n{}^i h^n{}_{jk} J_m{}^j h^{mk}{}_i = J_n{}^i J_m{}^j A^n{}_j{}^m{}_i \\ &= J_n{}^i J_m{}^j (A^n{}_j{}^m{}_i - A^n{}_i{}^m{}_j) + d^k{}_{kn} d^l{}_l{}^n \\ &= J_n{}^i J_m{}^j (A^n{}_j{}^m{}_i - A^n{}_i{}^m{}_j) + \frac{1}{4p^2}|\nabla p|^2. \end{aligned}$$

This and $\widetilde{\Delta}p = \Delta p + \langle \lambda, \nabla p\rangle$ gives (5.5). ■

5.1. **The lowest dimensional case,** $n = 2$.
The evolution equation for $p$ simplifies in case $n = 2$ in such a way that we are able to use this quantity to bound the second fundamental form and to prove longtime existence and convergence. Therefore, let $(L, J, \sigma)$ and $(M, K, \eta)$ be two Riemann surfaces of constant scalar curvature $S$. Then $(L \times M, (J, -K), \sigma \times \eta)$ is a Kähler-Einstein manifold with scalar curvature $\overline{R} = 2S$. The curvature tensors can be written as

$$\begin{aligned} S_{ijkl} &= \frac{S}{2}(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}), \\ T_{\alpha\beta\gamma\delta} &= \frac{S}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}). \end{aligned}$$

We also set

$$T_{ijkl} = T_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_l^\delta$$

and

$$\eta_{ij} = \eta_{\alpha\beta} f_i^\alpha f_j^\beta.$$

This implies the Lemma

**Lemma 5.2.** *For $n = 2$ we obtain*

$$\begin{aligned} \overline{R}_{ijkl} &= S_{ijkl} + T_{ijkl} = \frac{S}{2}\big(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk} + \eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}\big), \\ &= \frac{S}{p}\big(g_{ik}g_{jl} - g_{il}g_{jk}\big) \\ \overline{R}_{ijk\underline{l}} &= \frac{S}{2}\big((2\sigma_{ik} - g_{ik})\omega_{lj} - \omega_{li}(2\sigma_{jk} - g_{jk})\big), \\ \overline{R}^n{}_{jnl} &= S(\sigma_{jl} - g^{ik}\sigma_{il}\sigma_{jk}) = \frac{S}{p} \cdot g_{jl}, \\ \overline{R}^n{}_{jk\underline{n}} &= \frac{S}{2}\big(\omega_{jk} - 2\omega_{jl}g^{li}\sigma_{ik}\big), \\ \overline{R}_{ij} &= \frac{S}{2} \cdot g_{ij}, \\ \overline{\nabla}_E \overline{R}_{ABCD} &= 0. \end{aligned}$$

**Proof:** Let $\mu_1, \mu_2$ be the two eigenvalues of $\eta_{ij}\sigma^{jk}$. Then the two eigenvalues of $g_{ij}\sigma^{jk}$ are given by $1+\mu_1, 1+\mu_2$ and by (2.21) we must have $\mu_1\mu_2 = 1$. This implies the first, third and fifth equation. The second and fourth equation follow from $f^*\widetilde{\omega} = \omega$ and the last equation is trivial. ■

This lemma now gives

$$\begin{aligned} J_t{}^s \overline{R}^k{}_s{}^t{}_{\underline{k}} &= \frac{S}{2} J_t{}^s g^{tl}\left(\omega_{sl} - 2\omega_{sp} g^{pq}\sigma_{ql}\right) \\ &= \frac{S}{2} g^{tl}\left(-\sigma_{tl} + 2\sigma_{tp} g^{pq}\sigma_{ql}\right) \\ &= \frac{S}{2}\left(-1 + 2|\sigma_{ij}|^2\right) \end{aligned}$$

Now

$$\begin{aligned} |\sigma_{ij}|^2 &= \frac{1}{(1+\mu_1)^2} + \frac{1}{(1+\mu_2)^2} \\ &= \left(\frac{1}{1+\mu_1} + \frac{1}{1+\mu_2}\right)^2 - \frac{2}{(1+\mu_1)(1+\mu_2)} \\ &= 1 - \frac{2}{p} \end{aligned}$$

gives

$$J_t{}^s \overline{R}^k{}_s{}^t{}_{\underline{k}} = \frac{S}{2}\left(-1 + 2\left(1 - \frac{2}{p}\right)\right) = \frac{S}{2p}(p-4)$$

and

$$\begin{aligned} & 12\overline{R}^{njsk} A_{jkns} + 2\overline{R}^{ij} b_{ij} - 6\overline{R}^{ni}{}_n{}^j b_{ij} \\ = \;& 12\frac{S}{p}(g^{ns}g^{jk} - g^{nk}g^{js})A_{jkns} + S g^{ij} b_{ij} - 6\frac{S}{p} g^{ij} b_{ij} \\ = \;& 12\frac{S}{p}(|H|^2 - |A|^2) + S|A|^2 - 6\frac{S}{p}|A|^2 \\ = \;& S|A|^2 - 6\frac{S}{p}(3|A|^2 - 2|H|^2). \end{aligned}$$

In addition

$$\begin{aligned} A_{injs} - A_{ijns} &= \frac{|H|^2 - |A|^2}{2}\left(g_{in}g_{js} - g_{ij}g_{ns}\right), \\ J_n{}^i J_m{}^j (A^n{}_j{}^m{}_i - A^n{}_i{}^m{}_j) &= \frac{|H|^2 - |A|^2}{2} J_n{}^i J_m{}^j \left(\delta^n{}_j \delta^m{}_i - \delta^n{}_i \delta^m{}_j\right) \\ &= |A|^2 - |H|^2, \end{aligned}$$

where we have used $J_i{}^i = 0$. Then the evolution equations in the previous section simplify in 2 dimensions, namely

**Lemma 5.3.** *For $n = 2$ we obtain the evolution equations*

$$\begin{aligned}
\frac{\partial}{\partial t}\alpha &= \Delta\alpha + \langle\lambda, \nabla\alpha\rangle + \frac{S}{2}\alpha, \\
\frac{\partial}{\partial t}p &= \Delta p + \langle\lambda, \nabla p\rangle - \frac{3}{2p}|\nabla p|^2 - 2p(2|A|^2 - |H|^2) - S(p-4), \\
\frac{\partial}{\partial t}|H|^2 &= \Delta|H|^2 + \langle\lambda, \nabla|H|^2\rangle - 2|\nabla H|^2 + 2|a_{ij}|^2 + S(1 - \frac{2}{p})|H|^2, \\
\frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2 + \langle\lambda, \nabla|A|^2\rangle - 2|\nabla A|^2 + 2|b_{ij}|^2 + 2(|H|^2 - |A|^2)^2 \\
&+ S|A|^2 - 6\frac{S}{p}(3|A|^2 - 2|H|^2).
\end{aligned}$$

## 6. Results

Our idea is to use the evolution equation for $p$ to prove uniform estimates for $|A|^2$. To this end we will first need estimates for $|H|^2$ also. Let $k$ be a function depending on $p$ only and $m$ a function depending on $\alpha, t$. We denote its partial derivative with respect to $t$ with $m_t$ and the partial derivatives of $k$ w.r.t. $p$ resp. of $m$ w.r.t. $\alpha$ with primes. Then we define the function $f := mk|H|^2$ for functions $k$, $m$ to be determined. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}f &= m'k|H|^2\{\Delta\alpha + \langle\lambda, \nabla\alpha\rangle + \frac{S}{2}\alpha\} + m_t k|H|^2 \\
&+ km\{\Delta|H|^2 + \langle\lambda, \nabla|H|^2\rangle - 2|\nabla H|^2 + 2|a_{ij}|^2 + S(1 - \frac{2}{p})|H|^2\} \\
&+ k'm|H|^2\{\Delta p + \langle\lambda, \nabla p\rangle - \frac{3}{2p}|\nabla p|^2 - 2p(2|A|^2 - |H|^2) - S(p-4)\}
\end{aligned}$$

For the gradient and the Laplacian of $f$ we get

$$\begin{aligned}
\nabla f &= m'k|H|^2\nabla\alpha + mk'|H|^2\nabla p + mk\nabla|H|^2, \\
\Delta f &= m'k|H|^2\Delta\alpha + mk'|H|^2\Delta p + mk\Delta|H|^2 \\
&+ m''k|H|^4 + 2m'k'|H|^2\langle\nabla p, \nabla\alpha\rangle + 2m'k\langle\nabla|H|^2, \nabla\alpha\rangle \\
&+ mk''|H|^2|\nabla p|^2 + 2mk'\langle\nabla|H|^2, \nabla p\rangle
\end{aligned}$$

If we insert this into the preliminary evolution equation we get

$$\begin{aligned}
\frac{\partial}{\partial t}f &= \Delta f - m''k|H|^4 - 2m'k'|H|^2\langle\nabla p, \nabla\alpha\rangle - 2m'k\langle\nabla|H|^2, \nabla\alpha\rangle \\
&- 2mk'\langle\nabla|H|^2, \nabla p\rangle - mk''|H|^2|\nabla p|^2 + \langle\lambda, \nabla f\rangle \\
&+ \frac{S}{2}m'k|H|^2\alpha + m_t k|H|^2 - \frac{3k'm}{2p}|H|^2|\nabla p|^2 \\
&- 2k'mp|H|^2(2|A|^2 - |H|^2) - k'mS(p-4)|H|^2 \\
&- 2km|\nabla H|^2 + 2km|a_{ij}|^2 + Skm(1 - \frac{2}{p})|H|^2
\end{aligned}$$

Using the relation for the gradient we obtain

$$\begin{aligned}\frac{\partial}{\partial t}f &= \Delta f - m''k|H|^4 - 2m'k\langle\nabla|H|^2,\nabla\alpha\rangle\\ &- 2\frac{k'}{k}\langle\nabla f,\nabla p\rangle + 2\frac{m(k')^2}{k}|H|^2|\nabla p|^2\\ &- mk''|H|^2|\nabla p|^2 + \langle\lambda,\nabla f\rangle + \frac{S}{2}m'k|H|^2\alpha\\ &+ m_t k|H|^2 - \frac{3k'm}{2p}|H|^2|\nabla p|^2\\ &- 2k'mp|H|^2(2|A|^2-|H|^2) - k'mS(p-4)|H|^2\\ &- 2km|\nabla H|^2 + 2km|a_{ij}|^2 + Skm(1-\frac{2}{p})|H|^2.\end{aligned}$$

Now we have

$$\begin{aligned}&- 2km|\nabla H|^2 - 2m'k\langle\nabla|H|^2,\nabla\alpha\rangle\\ &= -2km(|\nabla H|^2 + 2\frac{m'}{m}\nabla_i H_j H^i H^j)\\ &= -2km|\nabla_i H_j + \frac{m'}{m}H_i H_j|^2 + 2k\frac{(m')^2}{m}|H|^4.\end{aligned}$$

Inserting gives

$$\begin{aligned}\frac{\partial}{\partial t}f &= \Delta f + \langle\lambda,\nabla f\rangle - 2\frac{k'}{k}\langle\nabla p,\nabla f\rangle - 2km|\nabla_i H_j + \frac{m'}{m}H_i H_j|^2\\ &- f|H|^2(\frac{m''}{m} - 2\frac{(m')^2}{m^2}) - (\frac{k''}{k} - 2\frac{(k')^2}{k^2} + \frac{3k'}{2kp})f|\nabla p|^2\\ &- f(\frac{2k'p}{k}(2|A|^2-|H|^2) - 2\frac{|a_{ij}|^2}{|H|^2})\\ &+ Sf(\frac{m'\alpha}{2m} + \frac{m_t}{Sm} - \frac{k'}{k}(p-4) + 1 - \frac{2}{p}).\end{aligned}$$

Now we choose $k = \sqrt{p}$ and $m = \frac{\sigma}{\cos(\sigma\alpha)}$ with $\sigma(t) = e^{-\frac{S}{2}t}$. This is possible for $t > 0$, if the angle function $\alpha$ is chosen as in Lemma 3.5 and under the assumption that $\operatorname{osc}(\alpha) \leq \pi$, because then the strong maximum principle implies $\cos(\sigma\alpha) > 0$ for $t > 0$. Then

$$\begin{aligned}
m_t &= -\frac{S}{2}m\big(1 + \alpha\sigma\tan(\sigma\alpha)\big),\\
m' &= \sigma m\tan(\sigma\alpha),\\
k' &= \frac{1}{2\sqrt{p}},\\
0 &= \frac{k''}{k} - 2\frac{(k')^2}{k^2} + \frac{3k'}{2kp},\\
0 &= \frac{m'\alpha}{2m} + \frac{m_t}{Sm} - \frac{k'}{k}(p-4) + 1 - \frac{2}{p},\\
\sigma^2 &= \frac{m''}{m} - 2\frac{(m')^2}{m^2}
\end{aligned}$$

so that in this case we obtain the equation

$$\begin{aligned}
\frac{\partial}{\partial t}f &= \Delta f + \langle\lambda, \nabla f\rangle - \frac{1}{p}\langle\nabla p, \nabla f\rangle - 2\sqrt{p}m|\nabla_i H_j + \frac{m'}{m}H_iH_j|^2\\
&- f((\sigma^2-1)|H|^2 + 2|A|^2 - 2\frac{|a_{ij}|^2}{|H|^2})
\end{aligned}$$

But since $|a_{ij}|^2 = b^{ij}H_iH_j \le |A|^2|H|^2$ we obtain the inequality

$$\frac{\partial}{\partial t}f \le \Delta f + \langle\lambda, \nabla f\rangle - \frac{1}{p}\langle\nabla p, \nabla f\rangle - (\sigma^2-1)f|H|^2$$

But if $S \le 0$ we see that $\sigma \ge 1$ and consequently

**Corollary 6.1.** *Assume $S \le 0$ and $osc(\alpha) \le \pi$ at $t = 0$. Then there exists a universal constant $c$ such that*

$$|H|^2 \le \frac{c}{\sqrt{p}}e^{\frac{S}{2}t}.$$

**Proof:** From the strong parabolic maximum principle and the evolution equation for $\alpha$ we deduce that $\cos(\sigma\alpha) > 0$ for $t > 0$ and then we can apply the maximum principle to $f$. ∎

Now we use the evolution equation for $p$ together with the maximum principle and also obtain

**Lemma 6.2.** *The function $(p-4)e^{St}$ is bounded above by its initial maximum. Consequently there exists a constant $c$ such that*

$$p \le 4 + ce^{-St}$$

*and the Lagrangian graphs generated by our symplectic maps stay graphs for all times.*

**Proof:** From

$$|h_{ijk} - c(H_ig_{jk} + H_jg_{ki} + H_kg_{ij})|^2 \ge 0$$

we obtain, choosing $c = \frac{1}{n+2}$, the inequality

$$|A|^2 \geq \frac{3}{n+2}|H|^2$$

for any dimension $n$ and on any Lagrangian submanifold. Here we have $n = 2$ so that $2|A|^2 - |H|^2 \geq 0$. Moreover $p \geq 4$ holds because by (2.21) $p = \lambda_1\lambda_2 = \lambda_1 + \lambda_2$, where $\lambda_i$ are the eigenvalues of the tensor $g_{ik}\sigma^{kj}$. Then the lemma is a direct consequence of the maximum principle applied to the evolution equation of $(p-4)e^{St}$. ∎

If we combine the last Lemma with Corollary 6.1 then we get

**Corollary 6.3.** *If $S \leq 0$ and $osc(\alpha) \leq \pi$ at $t = 0$, we obtain the estimate*

$$p|H|^2 \leq c$$

*for a universal constant $c$.*

Now define a new function $f := p|A|^2$. We want to prove that $f$ admits a uniform upper bound. We use the evolution equations for $p$ and $|A|^2$ to compute

$$\begin{aligned}
\frac{\partial}{\partial t}f &= |A|^2\{\Delta p + \langle\lambda, \nabla p\rangle - \frac{3}{2p}|\nabla p|^2 - 2p(2|A|^2 - |H|^2) - S(p-4)\} \\
&+ p\{\Delta|A|^2 + \langle\lambda, \nabla|A|^2\rangle - 2|\nabla A|^2 + 2|b_{ij}|^2 \\
&+ 2(|H|^2 - |A|^2)^2 + S|A|^2 - \frac{6S}{p}(3|A|^2 - 2|H|^2)\} \\
&= \Delta f - 2\langle\nabla|A|^2, \nabla p\rangle + \langle\lambda, \nabla f\rangle - 2p|\nabla A|^2 - \frac{3}{2p}|A|^2|\nabla p|^2 \\
&- 2f(2|A|^2 - |H|^2) + 4S|A|^2 + 2p|b_{ij}|^2 \\
&+ 2p(|H|^2 - |A|^2)^2 - 6S(3|A|^2 - 2|H|^2).
\end{aligned}$$

From

$$|h_{ijk}\nabla_s h_{lmn} - h_{lmn}\nabla_s h_{ijk}|^2 \geq 0$$

it follows that

$$|A|^2|\nabla A|^2 \geq \frac{1}{4}|\nabla|A|^2|^2$$

and then

$$-2\langle\nabla|A|^2, \nabla p\rangle - 2p|\nabla A|^2 \leq -\frac{1}{2|A|^2}\langle\nabla f, \nabla|A|^2\rangle - \frac{3}{2p}\langle\nabla f, \nabla p\rangle + \frac{3}{2p}|A|^2|\nabla p|^2$$

when $|A|^2 \neq 0$. Inserting this into the preliminary evolution equation for $f$ gives

$$\begin{aligned}
\frac{\partial}{\partial t}f &\leq \Delta f + \langle\lambda, \nabla f\rangle - \frac{1}{2|A|^2}\langle\nabla f, \nabla|A|^2\rangle - \frac{3}{2p}\langle\nabla f, \nabla p\rangle \\
&- 2f(2|A|^2 - |H|^2) + 2p|b_{ij}|^2 + 2p(|H|^2 - |A|^2)^2 \\
&- 6S(3|A|^2 - 2|H|^2) + 4S|A|^2
\end{aligned}$$

at those point where $|A|^2 \neq 0$. Now we observe that

$$\begin{aligned} |b_{ij}|^2 + (|A|^2 - |H|^2)^2 &= (b^{ij} - a^{ij})(b_{ij} + a_{ij}) + |a_{ij}|^2 + (|A|^2 - |H|^2)^2 \\ &= \frac{1}{2}(|A|^4 - |H|^4) + |a_{ij}|^2 + |A|^4 - 2|A|^2|H|^2 + |H|^4 \\ &\leq \frac{3}{2}|A|^4 - \frac{1}{3}|A|^2|H|^2 \end{aligned}$$

where we again used the fact that $|A|^2 \geq \frac{3}{n+2}|H|^2 = \frac{3}{4}|H|^2$ and $|a_{ij}|^2 = b^{ij}H_iH_j \leq |A|^2|H|^2$. We use this inequality to proceed with the estimate for $\frac{\partial}{\partial t}f$. Then

$$\begin{aligned} \frac{\partial}{\partial t}f &\leq \Delta f + \langle \lambda, \nabla f\rangle - \frac{1}{2|A|^2}\langle \nabla f, \nabla |A|^2\rangle - \frac{3}{2p}\langle \nabla f, \nabla p\rangle \\ &- 2f(2|A|^2 - |H|^2) + 2f(\frac{3}{2}|A|^2 - \frac{1}{3}|H|^2) \\ &- 6S(3|A|^2 - 2|H|^2) + 4S|A|^2 \\ &= \Delta f + \langle \lambda, \nabla f\rangle - \frac{1}{2|A|^2}\langle \nabla f, \nabla |A|^2\rangle - \frac{3}{2p}\langle \nabla f, \nabla p\rangle \\ &- f(|A|^2 - \frac{4}{3}|H|^2) - 14S|A|^2 + 12S|H|^2 \end{aligned}$$

and since $|H|^2 \leq \frac{4}{3}|A|^2$ and $f = |A|^2 p$ we can find a universal positive constant $c$ such that

$$\begin{aligned} \frac{\partial}{\partial t}f &\leq \Delta f + \langle \lambda, \nabla f\rangle - \frac{1}{2|A|^2}\langle \nabla f, \nabla |A|^2\rangle - \frac{3}{2p}\langle \nabla f, \nabla p\rangle \\ &- \frac{f}{p}(f - \frac{4}{3}p|H|^2 - c) \end{aligned}$$

holds whenever $|A|^2 \neq 0$. In the case $S \leq 0$ and $\text{osc}(\alpha) \leq \pi$ at $t = 0$ we can already estimate $p|H|^2$ from above by a universal constant also. Thus in this case

$$\begin{aligned} \frac{\partial}{\partial t}f &\leq \Delta f + \langle \lambda, \nabla f\rangle - \frac{1}{2|A|^2}\langle \nabla f, \nabla |A|^2\rangle - \frac{3}{2p}\langle \nabla f, \nabla p\rangle \\ &- \frac{f}{p}(f - \widetilde{c}). \end{aligned}$$

This implies that a positive increasing maximum of $f$ must be bounded by $\widetilde{c}$ and therefore

**Corollary 6.4.** *Assume $S \leq 0$ and osc$(\alpha) \leq \pi$ at $t = 0$. Then there exists a universal constant $c$ such that*

$$p|A|^2 \leq c$$

*holds for all $t \in [0, T)$. Consequently $|A|^2 \leq p|A|^2 \leq c$ and $T = \infty$.*

We are now able to prove the main theorem. By Corollary 6.4 we know that $T = \infty$ and $|A|^2$ is uniformly bounded from above. In case $S = 0$ we can use the bound for $p = \det(\sigma^{ij} g_{jk})$ to see that the induced metrics are all equivalent. In the other cases we have $S < 0$ and by Corollary 6.1 we see that there exist constants $c, \epsilon$ such that

$$|H|^2 \leq ce^{-\epsilon t}.$$

Then we use Lemma 8.2 in [3]

**Lemma 6.5.** *Let $g_{ij}$ be a time dependent metric on a compact manifold $L$ for $0 \leq t < T \leq \infty$. Suppose*

$$\int_0^T \max_L |\frac{\partial}{\partial t} g_{ij}| dt \leq C < \infty.$$

*Then the metrics $g_{ij}(t)$ for all different times are equivalent and they converge as $t \to T$ uniformly to a positive definite metric tensor $g_{ij}(T)$ which is continous and also equivalent. Here $|\frac{\partial}{\partial t} g_{ij}|^2 = g^{ij} g^{kl} \frac{\partial}{\partial t} g_{ik} \frac{\partial}{\partial t} g_{jl}$.*

We apply this Lemma to the original flow (1.1), i.e. to $\frac{\partial}{\partial t} F = \overrightarrow{H}$, which up to tangential deformations is the same flow as before. In particular all the estimates for intrinsic quantities are valid as well since there is no change in the time scale. This Lemma can be applied since $|\frac{\partial}{\partial t} g_{ij}|^2 = 4|a_{ij}|^2 \leq 4|A|^2|H|^2 \leq c|H|^2$ and the exponential estimate for $|H|^2$ and $T = \infty$ imply that the integral in this Lemma is bounded. The main theorem then follows from Theorem 1.2 (c).

*Remark* 6.6. In the same way we can now also prove the results stated in remark 1.9. We only have to take into account the evolution equation for $r$ defined in section 3.1 and the fact that $\widetilde{p} := \frac{4}{r^2}$ satisfies the same evolution equation as $p$ in the previous section (without the tangential movement induced by $\lambda$, i.e. for $\widetilde{\lambda} = 0$ in equation 3.1). Then the same estimates hold in this case as well, provided $\cos(\alpha) \geq 0$ at $t = 0$.

## 7. Appendix

Here we briefly discuss the 1-dimensional case, i.e. curves on Riemann surfaces with constant nonpositive curvature $\overline{R}$ and with initial oscillation of the Lagrangian angle $\alpha$ being smaller than $\pi$. The evolution equations induced by the main equation 1.1 simplify to

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \Delta\alpha + \frac{\overline{R}}{2}\alpha, \\ \frac{\partial}{\partial t}|H|^2 &= \Delta|H|^2 - 2|\nabla H|^2 + 2|H|^2(|H|^2 + \frac{\overline{R}}{2}). \end{aligned}$$

The parabolic maximum principle shows that $e^{-\frac{\overline{R}}{2}t}\alpha$ will stay bounded by its initial maximum and minimum (and it will be even slightly below resp. above by the strong parabolic maximum principle). Then for $\overline{R} \leq 0$ we may define the function $f := \frac{\sigma^2}{\cos^2(\sigma\alpha)}|H|^2$, with $\sigma(t) = e^{-\frac{\overline{R}}{2}t}$. A short calculation shows that the evolution equation of $f$ is given by

$$\frac{\partial}{\partial t}f = \Delta f + \langle \nabla f, X \rangle$$

for some tangent vector field $X$ if $|H|^2 \neq 0$. But then the parabolic maximum principle implies that $f \leq c$ for a universal constant $c$. In particular $|A|^2 = |H|^2 \leq c$. If $\frac{\overline{R}}{2} < 0$, then we may again apply Lemma 6.5 to prove that all metrics are uniformly equivalent. In case $\overline{R} = 0$ we observe that the problem can be reduced to curves in $\mathbb{R}^2$ and then the condition $\operatorname{osc}(\alpha) < \pi$ means that the curve can be written as an entire graph over one of its tangent lines. It is well known [2] that these curves will stay graphs and that $\cos(\alpha)$ corresponds to the induced line element. Consequently the bound for $\alpha$ proves that the metrics are also uniformly equivalent in the case $\overline{R} = 0$. Using Theorem 1.2 (c) we are done.

## References


[1] Cao, H.-D.: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.* **81**, *359-372 (1985).*

[2] Ecker, K.; Huisken, G.: Mean curvature evolution of entire graphs. *Ann. of Math. (2)* **130**, *453–471 (1989)*

[3] Huisken, G.: Flow by mean curvature of convex surfaces into spheres. *J. Differ. Geom.* **20**, *237-266 (1984).*

[4] Huisken, G.: Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. *Invent. Math.* **84**, *463-480 (1986).*

[5] Smoczyk, K: Harnack inequality for the Lagrangian mean curvature flow. *Calc. Var. Partial Differ. Equ.* **8, No.3**, *247-258 (1999).*

[6] Smoczyk, K: Der Lagrangesche mittlere Kruemmungsfluss. (The Lagrangian mean curvature flow). (German) Leipzig: Univ. Leipzig (Habil.-Schr.), 102 S. (2000)

[7] Strominger, A.; Yau, S.-T.; Zaslow, E.: Mirror symmetry is $T$-duality. *Nucl. Phys.*, **B 479, No.1-2**, *243-259 (1996).*

[8] Thomas, R. P.; Yau, S.-T.: More moment maps, monodromy and mirror manifolds. *preprint.*

[9] Wang, M.-T.: *in preparation*

[10] Wolfson, J.G.: Minimal Lagrangian diffeomorphisms and the Monge-Ampère equation. *J. Differ. Geom.* **46**, *No.2, 335-373 (1997).*

[11] Yau, S.-T. (Editor): *Mirror Symmetry I, AMS/IP Studies in Advanced Mathematics* **9** *(1998), International Press.*


Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22–26, D-04103 Leipzig, Germany,