Prescribing the Maslov form of Lagrangian immersions

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Abstract. We formulate and apply a modified Lagrangian mean curvature flow to prescribe the Maslov form of Lagrangian immersions in $\mathbb{C}^n$. We prove longtime existence results and derive optimal results for curves.

Keywords: Lagrangian, mean curvature flow, Maslov form

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1. Introduction

Let $(\mathbb{R}^{2n}, \omega, J)$ be the euclidean space equipped with the standard symplectic structure ω and complex structure J. A submanifold L of real dimension n is called Lagrangian, if the symplectic form ω vanishes identically when restricted to the tangent bundle of L. This is equivalent to the statement that J gives an isomorphism between the tangent bundle of L (considered as a subspace of $T\mathbb{R}^{2n}$) and the normal bundle with respect to the standard inner product $\langle \cdot, \cdot \rangle$. If L is Lagrangian and V a normal vector field, then

$$\eta_V := \omega(\cdot, V)$$

defines an associated 1-form on L. In particular if $V = \overrightarrow{H}$ is the mean curvature vector field, then $H = \eta_{\overrightarrow{H}}$ is called the mean curvature form of L. From the Codazzi equations it follows that H is closed and defines a cohomology class. It has been shown by Morvan (Morvan, 1981) that – up to a constant factor – H equals the Maslov form M of L (for the definition of M see (Morvan, 1981)). This relation is similar to the correspondence between the Ricci form of a Kähler manifold and its first Chern form. In (Cao, 1985) a modified Kähler-Ricci flow has been used to deform a given Kähler metric to a new Kähler metric with prescribed Ricci form. On the other hand there are several analogies between the mean curvature flow and the Ricci flow. We will therefore try to deform a given Lagrangian submanifold L_0 in $\mathbb{R}^{2n}$ with mean curvature form H_0 into a new Lagrangian submanifold with prescribed Maslov form $m \in [H_0]$. More precisely, we study the following parabolic system

$$\frac{d}{dt}F = -g^{ij}(H_i - m_i)\nu_j, \tag{1}$$

where $F : L \to I\!R^{2n}$ is a smooth family of Lagrangian immersions of L with $L_0 = F(L, t = 0)$. Here, g^{ij} is the inverse of the induced metric $g = F^*\langle\cdot,\cdot\rangle$, $\nu_i = J(\frac{\partial F}{\partial x^i})$ and $H = H_i dx^i, m = m_i dx^i$ in local coordinates x^i for L (the summation convention is used) and we will always assume that $m \in [H_0]$, i.e. $m - H_0$ is exact. Equation (1) is a quasilinear parabolic system and for compact smooth L_0 always admits a smooth shorttime solution on a maximal time interval $[0, T)$. From the Codazzi equation (see below) it follows that the resulting immersions $L_s := F(L, t = s)$ are Lagrangian also. In addition, from the evolution equation (9) for $H - m$ we conclude that the flow is Hamiltonian, i.e. there exists a smooth function α depending on t such that $d\alpha = H - m$. A proof of the following result will be given below.

PROPOSITION 1.1. *Assume that L is compact and that (1) admits a solution on the maximal time interval* $[0, T)$. *If the norm of the second fundamental form* $|A|$ *stays uniformly bounded above on* $[0, T)$*, then* $T = \infty$ *and* L_t *converges in the* C^∞*-topology to a smooth Lagrangian limit immersion* L_∞ *with mean curvature form* $H = m$. *The rate of convergence is exponential.*

Minimal Lagrangian tori in Calabi-Yau manifolds are important (Strominger et al., 1996). In the study of minimal Lagrangian submanifolds some progress has been made in the last years (Schoen et al., 1999; Schoen et al., 2000). The Lagrangian mean curvature flow ($m = 0$) presumably is a good tool to obtain such tori from suitable given initial Lagrangian tori because the flow becomes stationary if $H = 0$ ($= m$). The hard part, according to Proposition 1.1, is to derive uniform bounds for the second fundamental form. In many similar problems, e.g. the Kähler-Ricci flow, one obtains these bounds from lower order bounds. Unfortunately a general principle to perform such estimates in the Lagrangian mean curvature flow has not been found yet. Therefore a careful study of lower dimensional cases seems to be necessary. This is the reason, why we will mainly restrict our attention to the case $n = 1$. Here, we can indeed find some useful lower order estimates that will imply longtime existence and convergence. We mention that the curve shortening flow has been studied by many authors (Abresch et al., 1986; Angenent, 1991; Gage, 1984; Gage et al., 1986; Grayson, 1989) (we only mention a few) but the modified curve shortening flow has never been investigated before. We will present the following general longtime existence results for the modified mean curvature flow for curves in $\mathbb{R}^2$.

THEOREM 1.2. *Assume that* γ_t *are smooth, closed curves evolving according to (1). If the induced metrics* g_t *on* γ_t *are uniformly equivalent to the initial metric* g_0 *and* $H_0 - m = d\alpha_0$ *with* $osc(\alpha_0) \leq \pi$*, then* $T = \infty$ *and the curves smoothly and exponentially converge to a limit curve* γ_∞ *with* $H = m$.

THEOREM 1.3. *Assume that* $r : S^1 \to \mathbb{R}$ *is a smooth positive function and* γ_0 *is the initial curve given by*

$$\begin{aligned} F &: S^1 \to \mathbb{R}^2 \\ F(\phi) &:= (r(\phi)\cos(k\phi), r(\phi)\sin(k\phi)), \end{aligned}$$

where k *is a positive integer. Further assume that* $f : S^1 \to \mathbb{R}$ *is a smooth function such that*

$$|\arctan(\frac{r'}{kr})| < \frac{\pi}{2} - osc\ f.$$

Then the modified curve shortening flow with $m = (f - k\phi)' d\phi$ *admits a smooth solution for all* $t \in [0, \infty)$ *and* γ_t *converges in the* C^∞*-topology to a smooth limit curve* γ_∞ *with curvature form* $H = m$.

THEOREM 1.4. *Assume* $u, f : \mathbb{R} \to \mathbb{R}$ *are smooth periodic functions (with the same period) such that the initial curve given by*

$$\begin{aligned} F &: \mathbb{R} \to \mathbb{R}^2 \\ F(x) &:= (x, u(x)), \end{aligned}$$

satisfies

$$|\arctan(u')| < \frac{\pi}{2} - osc\ f.$$

Then the modified curve shortening flow with $m = df$ *admits a smooth solution for all* $t \in [0, \infty)$ *and* γ_t *converges in the* C^∞*-topology to a smooth limit curve* γ_∞ *with curvature form* $H = m$.

Some pictures of these results can be found below.

Figure 1. $k = 2, f(\phi) = \arctan(\frac{1}{4}\cos(\phi))$. The left curve is γ_0 (a double covering of S^1) and the right one depicts γ_∞

Figure 2. $k = 3, f(\phi) = \arctan(\cos(10\phi))$

1.1. Preliminaries

Now assume that $F : L \to (I\!R^{2n}, \omega, J)$ is a Lagrangian immersion. If $(x^i)_{i=1,\ldots,n}$ are local coordinates for L, then we set

$$e_i := \frac{\partial F}{\partial x^i} \; ; \; \nu_i := Je_i.$$

By the Lagrangian condition ν_i is a normal vector for any $i = 1, \ldots, n$. The second fundamental form on L can be defined as

$$h_{ijk} := -\langle \nu_i, \overline{\nabla}_{e_j} e_k \rangle,$$

where $\overline{\nabla}$ denotes the flat connection on $I\!R^{2n}$. In local coordinates the induced metric $g = F^*\langle \cdot, \cdot \rangle$ and the mean curvature 1-form $H_i dx^i$ are given by

$$\begin{aligned} g_{ij} &= \langle e_i, e_j \rangle, \\ H_i &= g^{kl} h_{ikl}. \end{aligned}$$

We also introduce

$$\begin{aligned} A_{ijkl} &:= h_{ijn} h^n{}_{kl}, \\ a_{kl} &:= A_i{}^i{}_{kl} = H^n h_{nkl}, \\ b_{kl} &:= A_k{}^i{}_{li} = h_k{}^{ij} h_{ijl}. \end{aligned}$$

The second fundamental form satisfies the following relations:

$$\begin{aligned} \textbf{Symmetry} &: h_{ijk} = h_{jik} = h_{jki}, && (2) \\ \textbf{Gauss equation} &: R_{ijkl} = A_{ikjl} - A_{iljk}, && (3) \\ \textbf{Codazzi equation} &: \nabla_i h_{jkl} - \nabla_j h_{ikl} = 0, && (4) \\ \textbf{traced Codazzi equation} &: \nabla_k H_l - \nabla_l H_k = 0. && (5) \end{aligned}$$

Applying the rule for interchanging covariant derivatives and the Codazzi equation we obtain a Simons type identity

$$\begin{aligned} \textbf{Simons identity} : \nabla_i \nabla_j H_k &= \Delta h_{ijk} - a_i{}^s h_{sjk} && (6) \\ &+ b_i{}^s h_{sjk} + b_j{}^s h_{ski} + b_k{}^s h_{sij} - 2h_{in}{}^m h_{jm}{}^s h_{ks}{}^n \end{aligned}$$

Figure 3. $k = 2, f(\phi) = \arctan(\frac{1}{4}\cos(3\phi))$

In the sequel we will set $\theta := H - m$. The following evolution equations can be easily derived from (1) (compare with (Oh, 1990; Smoczyk, 1999)).

PROPOSITION 1.5.

$$\frac{d}{dt} g_{ij} = -2\theta^n h_{nij}, \tag{7}$$

$$\frac{d}{dt} h_{ijk} = \nabla_i \nabla_j \theta_k - \theta^n (A_{ijkn} + A_{kijn}), \tag{8}$$

$$\frac{d}{dt} \theta_i = \nabla_i \nabla^j \theta_j. \tag{9}$$

We will now prove the following useful Lemma

LEMMA 1.6. **(Representation formula)** *Assume that $L_t = F_t(L)$ is compact, orientable and evolves under the modified MCF (1) and let Δ_t be the Laplace-Beltrami operator of the induced metrics at time t. There exists a unique smooth family of functions α_t such that*

$$H_t - m = d\alpha_t$$

$$\frac{d}{dt}\alpha_t = \Delta_t \alpha_t$$

$$\min_L \alpha_0 = -\max_L \alpha_0$$

Proof. From the evolution equation for $\theta = H - m$ we conclude that there exist smooth functions $\widetilde{\alpha}_t$ such that $\theta_t = d\widetilde{\alpha}_t$ and

$$\frac{d}{dt}\widetilde{\alpha}_t - \Delta_t \widetilde{\alpha}_t = c(t)$$

with a smooth function $c(t)$ depending only on t. Now define

$$\alpha_t = \widetilde{\alpha}_t - \int_0^t c(\tau) d\tau - \frac{1}{2}(\max_L(\widetilde{\alpha}_0) + \min_L(\widetilde{\alpha}_0)).$$

This function has all desired properties and is unique.

Proof of Proposition 1.1. Following the techniques in (Huisken, 1985) we obtain uniform bounds for all higher covariant derivatives of the second fundamental form A, i.e. there are constants c_k independent of t such that $|\nabla^k A|^2 \leq c_k$. It follows that all induced metrics are uniformly

Figure 4. The "figure eight" curve

equivalent to the initial metric and that $T = \infty$. Then all assumptions in Cao's Harnack inequality (Cao, 1985) are satisfied and we can apply this theorem to the angle function α from the representation formula 1.6. Consequently there exist constants A, c such that

$$\operatorname{osc}(\alpha) \leq Ae^{-ct}.$$

The evolution equation $\frac{d}{dt}\alpha = \Delta\alpha$ also implies that the maximum of α is nonincreasing and that the minimum is nondecreasing. So there must be constants $\psi, \widetilde{A}, \widetilde{c}$ such that

$$|\alpha - \psi|^2 \leq \widetilde{A}e^{-\widetilde{c}t}.$$

If (L, g) is a compact Riemannian manifold then there exists a constant p depending on g such that for all C^2 functions f on L we have the well-known interpolation inequality

$$|\nabla f|^2 \leq p|f| \cdot |\nabla^2 f|$$

Now in our situation all metrics are uniformly equivalent and p can be chosen independent of t. Applying the interpolation inequality to $f = \alpha - \psi$ we obtain that all norms $|\nabla^k(\alpha - \psi)|^2$ must tend to zero exponentially. In particular the speed $|\theta|^2$ exponentially tends to zero. This and the bound for the second fundamental forms imply that there exists a time t_1 such that all subsequent immersions for $t \geq t_1$ must stay in a compact region around L_{t_1} and can be represented as gradient graphs over L_{t_1}. The time derivative of the generating function then tends to zero with an exponential rate also. This proves the proposition.

2. Curves in $\mathbb{R}^2$

Let us first consider the figure eight curve

$$\begin{aligned} f &: S^1 \to \mathbb{R}^2 \\ f(\phi) &:= \frac{1}{1 + \sin^2(\phi)}(\cos(\phi), \cos(\phi)\sin(\phi)). \end{aligned}$$

The (mean) curvature form is exact. But the modified curve shortening flow $\frac{d}{dt}F = (m^i - H^i)\nu_i$ cannot have a smooth solution converging to a loop with $H_i = m_i$ for all m with $[m - H] = 0$ since this obviously is wrong for $m = 0$. Therefore the modified mean curvature flow is obstructed even in the lowest dimensional case. Let us assume that $\gamma \in I\!R^2$ is a smooth curve given by the map

$$\begin{aligned} F &: S^1 \to I\!R^2 \\ F(\phi) &:= (r(\phi)\cos(k\phi), r(\phi)\sin(k\phi)), \end{aligned}$$

where r is a positive smooth function on S^1 and k is a positive integer. We abbreviate a derivative w.r.t. ϕ by a prime. Then

$$e := F' = (r'\cos(k\phi) - kr\sin(k\phi), r'\sin(k\phi) + kr\cos(k\phi)), \tag{10}$$

$$\nu := Je = (-r'\sin(k\phi) - kr\cos(k\phi), r'\cos(k\phi) - kr\sin(k\phi)), \tag{11}$$

$$g = \langle e, e\rangle = (kr)^2 + (r')^2. \tag{12}$$

Note that ν is inward pointing. For the (mean) curvature form we get

$$\begin{aligned} H &= -\frac{1}{g}\langle F'', \nu\rangle d\phi \\ &= \frac{1}{g}\big((r'' - k^2 r)\cos(k\phi) - 2kr'\sin(k\phi)\big)\big(kr\cos(k\phi) + r'\sin(k\phi)\big)d\phi \\ &+ \big(2kr'\cos(k\phi) + (r'' - k^2 r)\sin(k\phi)\big)\big(-r'\cos(k\phi) + kr\sin(k\phi)\big)d\phi \\ &= \frac{kr(r'' - k^2 r) - 2k(r')^2}{(kr)^2 + (r')^2}d\phi \\ &= (\arctan(\frac{r'}{kr}) - k\phi)'d\phi. \end{aligned} \tag{13}$$

The Lagrangian angle $\widetilde{\beta}$, i.e. the multivalued function $\widetilde{\beta}$ with $H = d\widetilde{\beta}$, is therefore given by

$$\widetilde{\beta} = \arctan(\frac{r'}{kr}) - k\phi. \tag{14}$$

Since $\arctan(\frac{r'}{kr})$ is a function on S^1 we see that

$$\int_{S^1} H = -2k\pi.$$

Hence k determines the cohomology class of H. A 1-form m on S^1 lies in the same cohomology class as H if and only if

$$m = (f - k\phi)'d\phi, \tag{15}$$

where f is a smooth function on S^1. Let us now assume that γ_t is a smooth family of curves evolving by the modified curve shortening flow

$\frac{d}{dt}F = (m^i - H^i)\nu_i$ with m_i defined as in (15). Further assume that they can be written in the form above, i.e. as graphs over S^1. From

$$\frac{d}{dt}F = (m^i - H^i)\nu_i = \frac{(f - \arctan(\frac{r'}{kr}))'}{(kr)^2 + (r')^2}\nu$$

and $\frac{d}{dt}r = \frac{\partial}{\partial t}r + r'\frac{d}{dt}\phi$ we obtain

$$\frac{\partial}{\partial t}r = \frac{\beta'}{kr} = \frac{1}{(kr)^2 + (r')^2}(krr'' - k(r')^2) - \frac{f'}{kr}, \tag{16}$$

where we set

$$\beta := \arctan(\frac{r'}{kr}) - f. \tag{17}$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}\beta &= \frac{1}{1 + (\frac{r'}{kr})^2}\frac{1}{(kr)^2}(kr(\frac{\beta'}{kr})' - \beta'\frac{r'}{r}) \\ &= \frac{1}{(kr)^2 + (r')^2}(\beta'' - 2\frac{r'}{r}\beta').\end{aligned} \tag{18}$$

We want to prove that the parabolic equation (16) admits an immortal smooth solution such that r tends to a smooth positive limit function as $t \to \infty$, provided the function f is chosen suitably. Figures (1) - (3) describe the evolution of the k-time covered unit circle (left picture for $t = 0$) by the modified curve shortening flow to one with prescribed curvature form $H = (f - k\phi)'d\phi$ (right picture for $t = \infty$). We begin with first and zero order estimates for r.

LEMMA 2.1. *Assume that (16) admits a smooth positive solution r on a time interval $[0, T)$ and that for $t = 0$ we have*

$$|\arctan(\frac{r'}{kr})| < \frac{\pi}{2} - osc\ f. \tag{19}$$

Then there exist constants $c_1, c_2, c_3 > 0$ independent of T such that

$$c_1 < r < c_2, \quad |r'| < c_3.$$

Proof. From the evolution equation (18) for the function β and the parabolic maximum principle we obtain that

$$\min_{t=0}\beta \le \beta(\phi, t) \le \max_{t=0}\beta, \forall t \in [0, T).$$

Consequently

$$\arctan(\frac{r'}{kr})(\phi, t) \le \max_{t=0}\beta + \max f \le \max_{t=0}\big(\arctan(\frac{r'}{kr})\big) + \text{osc } f < \frac{\pi}{2}.$$

Similarly

$$\arctan(\frac{r'}{kr}) > -\frac{\pi}{2}.$$

This shows that $|\frac{r'}{kr}|$ is bounded above, i.e. there exists a constant c_4 independent of T such that

$$(r')^2 \leq c_4 r^2.$$

Then we can use (16) to obtain

$$\frac{\partial}{\partial t}\int_{[0,2\pi)} kr^2 d\phi = 2\int_{[0,2\pi)} \beta' d\phi = 0$$

and observe that for any $t \in [0,T)$ there exists a point $p \in S^1$ such that $r^2(p,t) = \frac{1}{2\pi}\int_{[0,2\pi)} r^2(\phi,0)d\phi$. This together with the gradient estimate for $\log(r)$ gives the result.

COROLLARY 2.2. *Under the assumptions in Lemma 2.1 the induced metrics $g = (kr)^2 + (r')^2$ are uniformly equivalent to the initial metric for all $t \in [0,T)$.*

Proof of Theorem 1.2. The evolution equation for $H - m$ is given by

$$\frac{d}{dt}(H-m) = dd^\dagger(H-m).$$

Choose α as in the representation formula 1.6. From the strong parabolic maximum principle and the evolution equation for α we observe that $|\alpha| < \frac{\pi}{2}$. for all $t > 0$. Consequently, for all $\epsilon \in (0,T)$ we can find a constant $\sigma > 1$ depending on ϵ such that $f := \cos(\sigma\alpha)$ is positive for all $t \in (\epsilon, T)$. Let us now compute the evolution equation for $\frac{|\nabla\alpha|^2}{f^2}$ which makes sense for $t \geq \epsilon$. We have

$$\begin{aligned}\frac{d}{dt}\frac{|\nabla\alpha|^2}{f^2} &= -2\frac{|\nabla\alpha|^2}{f^3}(\Delta f - f''|\nabla\alpha|^2)\\ &+ \frac{1}{f^2}(2\nabla^i\alpha h_i^{\ jk}\nabla_j\alpha\nabla_k\alpha + 2\nabla^i\alpha\nabla_i(\Delta\alpha))\end{aligned}$$

If we take into account that the dimension is 1, then we see that

$$\begin{aligned}\nabla^i\alpha h_i^{\ jk}\nabla_j\alpha\nabla_k\alpha &= \langle d\alpha, H\rangle|\nabla\alpha|^2 = |\nabla\alpha|^4 + \langle d\alpha, m\rangle|\nabla\alpha|^2,\\ 2\nabla^i\alpha\nabla_i(\Delta\alpha) &= \Delta|\nabla\alpha|^2 - 2|\nabla^2\alpha|^2.\end{aligned}$$

On the other hand

$$\Delta(\frac{|\nabla\alpha|^2}{f^2}) = \frac{1}{f^2}\Delta|\nabla\alpha|^2 - 2\frac{|\nabla\alpha|^2}{f^3}\Delta f - \frac{4}{f}\langle\nabla f, \nabla\frac{|\nabla\alpha|^2}{f^2}\rangle - 2\frac{(f')^2}{f^4}|\nabla\alpha|^4.$$

This gives

$$\begin{aligned}\frac{d}{dt}\frac{|\nabla\alpha|^2}{f^2} &= \Delta(\frac{|\nabla\alpha|^2}{f^2}) + \frac{4}{f}\langle\nabla f, \nabla\frac{|\nabla\alpha|^2}{f^2}\rangle + 2\frac{(f')^2}{f^4}|\nabla\alpha|^4 \\ &- \frac{2}{f^2}|\nabla^2\alpha|^2 + \frac{2}{f^2}(|\nabla\alpha|^4 + \langle d\alpha, m\rangle|\nabla\alpha|^2) + \frac{2f''}{f^3}|\nabla\alpha|^4 \\ &= \Delta(\frac{|\nabla\alpha|^2}{f^2}) + \frac{4}{f}\langle\nabla f, \nabla\frac{|\nabla\alpha|^2}{f^2}\rangle - \frac{2}{f^2}|\nabla^2\alpha|^2 \\ &+ \frac{2}{f^2}(1 + \frac{(f')^2}{f^2} + \frac{f''}{f})|\nabla\alpha|^4 + \frac{2}{f^2}\langle d\alpha, m\rangle|\nabla\alpha|^2\end{aligned}$$

Fix $\epsilon > 0$ and assume that $(x, t_0), t_0 > \epsilon$ is a point where

$$0 < \frac{|\nabla\alpha|^2}{f^2} = \max_{\gamma\times[\epsilon,t_0]}(\frac{|\nabla\alpha|^2}{f^2}).$$

Then

$$\langle\nabla(\frac{|\nabla\alpha|^2}{f^2}), \nabla\alpha\rangle = 0 = \frac{2|\nabla\alpha|^2}{f^2}(\Delta\alpha - \frac{f'}{f}|\nabla\alpha|^2).$$

Therefore $\Delta\alpha = \frac{f'}{f}|\nabla\alpha|^2$ at (x, t_0). Since for $n = 1$ we have $(\Delta\alpha)^2 = |\nabla^2\alpha|^2$ we obtain at (x, t_0)

$$\begin{aligned}0 \le \frac{d}{dt}(\frac{|\nabla\alpha|^2}{f^2}) &\le \frac{2}{f^2}(1 + \frac{f''}{f})|\nabla\alpha|^4 + \frac{2}{f^2}\langle d\alpha, m\rangle|\nabla\alpha|^2 \\ &= \frac{2}{f^2}|\nabla\alpha|^2\big((1-\sigma^2)|\nabla\alpha|^2 + \langle d\alpha, m\rangle\big)\end{aligned}$$

Now we use the Cauchy-Schwarz inequality and $\sigma > 1$ to estimate

$$\begin{aligned}0 &\le (1-\sigma^2)|\nabla\alpha|^2 + \langle d\alpha, m\rangle \\ &\le \frac{1-\sigma^2}{2}|\nabla\alpha|^2 + \frac{1}{2(\sigma^2-1)}|m|^2.\end{aligned}$$

Since m is fixed and all metrics are uniformly equivalent by assumption, there exists a constant c such that

$$\frac{1-\sigma^2}{2}|\nabla\alpha|^2 + \frac{1}{2(\sigma^2-1)}|m|^2 \le \frac{1-\sigma^2}{2}|\nabla\alpha|^2 + c$$

and we must have

$$\frac{|\nabla\alpha|^2}{f^2} \le \frac{2c}{(\sigma^2-1)f^2} \le \widetilde{c}$$

for a new constant $\widetilde{c}$ which is independent of t (but depends on ϵ). Consequently $\frac{|\nabla\alpha|^2}{f^2}$ is uniformly bounded on (ϵ, T) and then also

$$|H|^2 = |d\alpha + m|^2 \le 2(|\nabla\alpha|^2 + \widetilde{c}_1) \le \widetilde{c}_2$$

for constants $\widetilde{c}_1, \widetilde{c}_2$ independent of t. This means that the assumptions in Proposition 1.1 are satisfied.

The proof of Theorem 1.3 is now a direct consequence of Theorem 1.2 and Corollary 2.2. To prove Theorem 1.4 we can proceed similarly. A short calculation shows that the modified curve shortening flow for graphs over the real axis is given by

$$\frac{\partial}{\partial t}u = \arctan(u') + f$$

and that

$$\frac{\partial^2 u}{\partial t^2} = \widetilde{\Delta}\frac{\partial}{\partial t}u$$

with $\widetilde{\Delta} = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$. The maximum principle implies that $\frac{\partial}{\partial t}u$ stays bounded above by its initial maximum and bounded from below by the initial minimum. Therefore

$$\arctan(u') < \frac{\pi}{2} - \text{osc } f + \max f - f \leq \frac{\pi}{2}$$

and also

$$\arctan(u') > -\frac{\pi}{2}$$

and these estimates are uniformly, i.e. $|u'|$ is uniformly bounded. This just means that the induced metrics $g = 1 + (u')^2$ stay uniformly equivalent and Theorem 1.4 is a consequence of Theorem 1.2.

REMARK . *The fact that all metrics are uniformly equivalent has only been used to estimate $|m|^2$. On the other hand, all results stated in this paper are also valid for Lagrangian submanifolds in flat Riemannian manifolds. In particular $m = 0$ makes sense for closed curves on flat surfaces because they are all covered by $\mathbb{R}^2$. The requirement $osc(\alpha) < \pi$ then means that the lifted curve in the cover space $\mathbb{R}^2$ can be represented as a periodic graph over one of its tangent lines. Moreover, the oscillation of the Lagrangian angle $\widetilde{\beta}$ for a closed curve in $\mathbb{R}^2$ is always bigger than π (figure eight curve). We also mention that we found a proof for a much weaker result that holds for the modified Lagrangian mean curvature flow of Lagrangian graphs over $\mathbb{R}^n \subset \mathbb{R}^{2n}$.*

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