SASAKI-RICCI FLOW

KNUT SMOCZYK, GUOFANG WANG, AND YONGBING ZHANG

1. Introduction

Recently Sasakian geometry, especially Sasaki-Einstein geometry, plays an important role in the Ads/CFT correspondence. The important problem in Sasakian geometry is to find Sasaki-Einstein metrics. Boyer, Galicki and their collaborators found many new Sasaki-Einstein metrics on quasi-regular Sasakian manifolds [2], [4] and [3]. The first class of irregular Sasaki-Einstein metrics was found by Gauntlett, Martelli, Spark and Waldram in [14] and [15]. For the study of Sasaki-Einstein manifolds, see also [20] and [21], [9], [19], [22], [23] and [13].

In this paper, we want to present a flow approach, as the Kähler-Ricci flow in Kähler geometry, to study the existence of Sasaki-Einstein metrics, more precisely, η-Einstein metrics on Sasakian manifold. The Sasakian geometry can be viewed as an odd-dimensional counterpart of the Kähler geometry. In fact, (M, g) is Sasakian if and only if its metric cone $(M \times \mathbb{R}_+, dr^2 + r^2 g)$ is Kähler. However, the Kähler-Ricci flow can not be applied directly to its metric cone, since the Kähler-Ricci flow does not preserve the cone structure.

To deform the Sasakian structures we will exploit the transverse structure of Sasakian manifolds as in [10] and [1]. On a Sasakian manifold, there is a natural foliation structure $\mathcal{F}_\xi$. In fact, it is a transverse Kähler structure. We want to deform the transverse Kähler structures by using a *transverse Kähler-Ricci flow*, which we call the *Sasaki-Ricci* flow. See its definition in Section 3. When the Sasakian manifold M is regular, its base space of the foliation is a Kähler manifold. In this case the Sasaki-Ricci flow can be reduced to the Kähler-Ricci flow on the base space. However, in general the base space is very wild and has not any manifold structure.

In this paper we will prove the well-posedness of Sasaki-Ricci flow and global existence. We will show that the Sasaki-Ricci flow converges to an η-Einstein metric provided that its basic first Chern class is negative or null. The existence of an η-Einstein metric in these cases, together with the *transverse Yau theorem*, were proved in [10] by the continuity method. See also [1]. In other words, we present an odd-dimensional counterpart of Cao's result [8] for the Kähler-Ricci flow. When the basic first Chern class is positive, in general one can not expect the convergence of the Sasaki-Ricci flow. In this case a natural object, the Sasaki-Ricci soliton, arises. We speculate that in this case the Sasaki-Ricci flow will converge to a Sasaki-Ricci soliton in a suitable sense, like the Kähler-Ricci flow [29], which was inspired by Perelman's work [25] (see also [26].). There are many works about the Kähler-Ricci flow, after Cao's work [8]. Sasaki-Ricci soliton plays a crucial role in [13] to find new Sasaki-Einstein metrics.

Our work is inspired by the *transverse Ricci flow* introduced by Lovrić, Min-Oo and Ruh [18] in transverse Riemannian geometry, which is an analogue of the Ricci flow introduced by Hamilton [17] that plays a crucial role in the resolution of the Poincarè conjecture [24] and [25]. In fact, after showing the well-posedness of the Sasaki-Ricci flow, we know that these two flows are equivalent.

This is a project supported by "Global Methods in Complex Geometry" Priority Programme of German Research Foundation, SPP 1094.

2. Sasakian manifolds

Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection of the Riemannian metric g, and let $R(X, Y)$ denote the Riemann curvature tensor of ∇.

By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. In particular $\eta \wedge (d\eta)^n$ defines a volume element on M, and hence M is orientable. There is a canonical vector field ξ defined by

$$\eta(\xi) = 1 \quad \text{ and } d\eta(\xi, X) = 0,$$

for any vector field X. ξ is called the *characteristic vector field* or *Reeb vector field.* η defines a $2n$-dimensional vector bundle $\mathcal{D}$ over M, where at each point $p \in M$ the fiber $\mathcal{D}_p$ of $\mathcal{D}$ is given by

$$\mathcal{D}_p = \mathrm{ker}\eta_p.$$

There is a decomposition of the tangential bundle TM

$$TM = \mathcal{D} \oplus L\xi,$$

where $L\xi$ is the trivial bundle generated by the Reeb field ξ. A contact manifold with a Riemannian metric g and a tensor field Φ of type (1,1) satisfying

$$\Phi^2 = -I + \eta \otimes \xi$$

and

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

is called an *almost metric contact manifold.* Such an almost metric contact manifold is called *Sasakian* if one of the following equivalent conditions holds:

(1) There exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1, 1)$, defined by $\Phi(X) = \nabla_X \xi$, satisfies the condition

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi$$

for any pair of vector fields X and Y on M.

(2) There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition
$$R(X,\xi)Y \;=\; g(\xi,Y)X - g(X,Y)\xi,$$
for any pair of vector fields X and Y on M.
(3) There exists a Killing vector field ξ of unit length on M so that the sectional curvature of every section containing ξ equals one.
(4) The metric cone $(C(M),\bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ is Kähler.

For other equivalent definitions and the proof of the equivalence, see [1]. By (4), a Sasakian manifold can be viewed as an odd dimensional counterpart of a Kähler manifold.

A Sasaki manifold (M,ξ,η,Φ,g) is a Sasaki-Einstein manifold if g is an Einstein metric, i.e.,
$$\mathrm{Ric}_g = cg,$$
for some constant c. Due to property (2) of the Sasakian manifold, it is easy to see that $c = 2n$. A Sasaki manifold (M,ξ,η,Φ,g) is an η-Einstein manifold if g satisfies
$$\mathrm{Ric}_g = \lambda g + \nu \eta \otimes \eta, \tag{2.1}$$
for some constant λ and ν. It is easy to see that $\lambda + \nu = 2n$. For a recent study of η-Einstein manifolds, see [4].

3. Transverse Kähler structures

In this section we exploit the transverse structure of Sasakian manifolds. In the sequel, we always assume that M is a Sasaki manifold with Sasakian Structure (ξ,η,g,Φ). To well understand the Sasakian structure, we consider the *transverse Kähler* structure on M. Let $\mathcal{F}_\xi$ is the *characteristic foliation* generated by ξ. On $\mathcal{D}$, it is naturally endowed with both a complex structure $\Phi_{|\mathcal{D}}$ and a symplectic structure $d\eta$. $(\mathcal{D},\Phi_{|\mathcal{D}},d\eta)$ gives M a transverse Kähler structure with Kähler form $d\eta$ and transverse metric g^T defined by
$$g^T(X,Y) = d\eta(X,\Phi Y).$$
The metric g^T is related to the Sasakian metric g by
$$g = g^T + \eta \otimes \eta.$$

In order to consider the deformations of Sasakian structures, we consider the quotient bundle of the foliation $\mathcal{F}_\xi$, $\nu(\mathcal{F}_\xi) = TM/L\xi$. There is an isomorphism between $\nu(\mathcal{F}_\xi)$ and $\mathcal{D}$. Let $p : TM \to \nu(\mathcal{F}_\xi)$ be the projection. g^T gives a bundle map $\sigma : \nu(\mathcal{F}_\xi) \to D$ which splits the exact sequence
$$0 \to L_\xi \to TM \to \nu(\mathcal{F}_\xi) \to 0,$$
i.e. $p\circ\sigma = id$. Denoting $\sigma^* g^T$ still by g^T, $\sigma : (\nu(\mathcal{F}_\xi), g^T) \to (\mathcal{D}, g^T)$ is a metric isomorphism. We might identify $\mathcal{D}$ with $\nu(\mathcal{F}_\xi)$. However, since we want to deform Sasakian metrics, it is better to distinguish them. In fact, under the deformations considered later, $\mathcal{D}$ changes, while $\nu(\mathcal{F}_\xi)$ keeps fixed.

But, for simplicity of notation, we will use the same notation and $\sigma = id$ if there is no confusion, especially if we do not consider deformations.

Let ∇ be the Levi-Civita connection associated to the Riemannian metric g on M. From the transverse metric g^T, one can define a transverse Levi-Civita connection on $\nu(\mathcal{F}_\xi)$ as follows

$$\nabla^T_X V = \begin{cases} (\nabla_X \sigma(V))^p, & \text{if } X \text{ is a section of } \mathcal{D}, \\ [\xi, \sigma(V)]^p, & \text{if } X = \xi, \end{cases} \tag{3.1}$$

where V is a section of $\nu(\mathcal{F}_\xi)$ and $X^p = p(X)$ the projection of X onto $\nu(\mathcal{F}_\xi)$. The connection ∇^T is called a *transverse Levi-Civita connection* on the normal bundle $\nu(\mathcal{F}_\xi)$, since this connection is torsion-free and metric. Namely it satisfies

$$\nabla^T_X Y - \nabla^T_Y X - [X, Y]^p = 0 \tag{3.2}$$

and

$$X g^T(V, W) = g^T(\nabla^T_X V, W) + g^T(V, \nabla^T_X W), \tag{3.3}$$

where $X, Y \in TM$ and $V, W \in \nu(\mathcal{F}_\xi)$. From this connection one can define the transverse curvature operator by

$$R^T(X, Y) = \nabla^T_X \nabla^T_Y - \nabla^T_Y \nabla^T_X - \nabla^T_{[X,Y]}$$

as usual. From the transverse curvature operator we define the transverse Ricci curvature by

$$\mathrm{Ric}^T(X, Y) = g(R^T(X, e_i)e_i, Y),$$

where e_i is an orthonormal basis of $\mathcal{D}$. We remark that here we have used the identification between $\mathcal{D}$ and $\nu(\mathcal{F}_\xi)$. More precisely the transverse Ricci tensor is defined by

$$\mathrm{Ric}^T(X, Y) = g(R^T(X, e_i)\sigma^{-1}(e_i), Y)$$

for $X, Y \in TM$.

One can check easily that

$$\mathrm{Ric}^T(X, Y) = \mathrm{Ric}(X, Y) + 2g^T(X, Y). \tag{3.4}$$

In the transverse Riemannian geometry, it is natural to ask if there exists a "best metric"–the transverse Einstein metric, which is a transverse metric satisfying the follow equation

$$\mathrm{Ric}^T = c g^T, \tag{3.5}$$

for certain constant c. (3.5) is a system of transverse elliptic equations [10]. A Sasakian metric is a transverse Einstein metric if and only if it is an η-Einstein metric. It is difficult to study the existence of (3.5) in general. (3.5) can be considered in a general setting–transverse Riemannian foliation. Analogous to the Ricci flow introduced by Hamilton, a transverse Ricci flow was introduced in [18] for a general foliation

$$\frac{d}{dt} g^T = -\mathrm{Ric}^T. \tag{3.6}$$

In [18], the short-time existence and the uniqueness of (3.6) was proved. Note that this is not a parabolic flow. This is a so-called transverse parabolic flow. Note that a transverse metric on the foliation $\nu(\mathcal{F}_\xi)$ need not to be a Sasakian metric. It is not easy to check directly if the flow (3.6) preserves the Sasakian structure (ξ, η, Φ, g). Instead, we will introduce another flow -which we call the Sasaki-Ricci flow, as an analogue of the Kähler-Ricci flow. Eventually, we will show that the Sasaki-Ricci flow is equivalent to (3.6) in the Sasakian setting.

In order to introduce the Sasaki-Ricci flow, we first consider deformations of Sasakian structures which preserve the Reeb field ξ, and hence the characteristic foliation $\mathcal{F}_\xi$. Such deformations were studied by Boyer and Galicki in [1].

A p-form α on M is called *basic* if

$$i(\xi)\alpha = 0, \quad \mathcal{L}_\xi \alpha = 0.$$

Let Λ^p_B be the sheaf of germs of basic p-forms and $\Omega^p_B = \Gamma(M, \Lambda^p_B)$ the set of all global section of Λ^p_B. A function f is basic if and only if $\xi(f) = 0$. It is easy to see that the exterior differential d preserves basic forms. Namely, if α is a basic form, so is $d\alpha$. Note that $d\eta$ is a basic 2-form, though η is not a basic form. The basic cohomology can be defined in a usual way.

We consider the complexified bundle $\mathcal{D}^{\mathbb{C}} = \mathcal{D} \otimes \mathbb{C}$. Using the structure Φ we decompose $\mathcal{D}^{\mathbb{C}}$ into two subbundles $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$, where $\mathcal{D}^{1,0} = \{X \in \mathcal{D}^{\mathbb{C}} \mid \Phi X = \sqrt{-1}X\}$ and $\mathcal{D}^{0,1} = \{X \in \mathcal{D}^{\mathbb{C}} \mid \Phi X = -\sqrt{-1}X\}$. Similarly, we decompose the complexified space $\Lambda^r_B \otimes \mathbb{C} = \oplus_{p+q=r} \Lambda^{p,q}_B$, where $\Lambda^{p,q}_B$ denotes the sheaf of germs of basic forms of type (p, q). Define ∂_B and $\bar{\partial}_B$ by

$$\begin{array}{rcl} \partial_B : \Lambda^{p,q}_B & \to & \Lambda^{p+1,q}_B \\ \bar{\partial}_B : \Lambda^{p,q}_B & \to & \Lambda^{p,q+1}_B, \end{array}$$

which is the decomposition of d. Let $d_B = d_{|\Omega^p_B}$. We have $d_B = \partial_B + \bar{\partial}_B$. Let $d^c_B = \frac{1}{2}\sqrt{-1}(\bar{\partial}_B - \partial_B)$. It is clear that

$$d_B d^c_B = i\partial_B \bar{\partial}_B, \qquad d^2_B = (d^c_B)^2 = 0.$$

Let $d^*_B : \Omega^{p+1}_B \to \Omega^p_B$ be the adjoint operator of $d_B : \Omega^p_B \to \Omega^{p+1}_B$. The basic Laplacian Δ_B is defined

$$\Delta_B = d^*_B d_B + d_B d^*_B.$$

Suppose that (ξ, η, Φ, g) defines a Sasakian structure on M. Let φ be a basic function. Put

$$\tilde{\eta} = \eta + d^c_B \varphi.$$

It is clear that

$$d\tilde{\eta} = d\eta + d_B d^c_B \varphi = d\eta + \sqrt{-1}\partial_B \bar{\partial}_B \varphi.$$

For small φ, $d\tilde{\eta}$ is non-degenerate in the sense that $\tilde{\eta} \wedge (d\tilde{\eta})^n \neq 0$. Set

$$\begin{array}{rcl} \widetilde{\Phi} & = & \Phi - \xi \otimes (d^c_B \varphi) \circ \Phi \\ \tilde{g} & = & d\tilde{\eta} \circ (Id \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}. \end{array}$$

Lemma 3.1. *$(M, \xi, \tilde{\eta}, \widetilde{\Phi}, \tilde{g})$ is also a Sasakian structure.*

Proof. See [4]. One can also check the Lemma by using the local frame given in the next section. □

Note that these deformations fix the Reeb field and change another three structures η, Φ, g and also $\mathcal{D}$. Now we consider $\nu(\mathcal{F}_\xi)$ instead of $\mathcal{D}$. On $\nu(\mathcal{F}_\xi)$ there is a complex structure $J : \nu(\mathcal{F}_\xi) \to \nu(\mathcal{F}_\xi)$ induced from Φ by

$$JV = (\Phi\sigma(V))^p$$

for any section of $\nu(\mathcal{F}_\xi)$. See also [6]. Recall that $\sigma : \nu(\mathcal{F}_\xi) \to \mathcal{D}$ is a bundle map splitting TM. One can check easily that J is independent of the deformations given above. On $\nu(\mathcal{F}_\xi)$, we can consider also its complexification and define the decomposition by using the complex structure J as above.

There are other kinds of deformations. For instance, the so-called $\mathcal{D}$-homothetic deformation [30] is defined

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\Phi} = \Phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta$$

for a positive constant a. Note that from a η-Einstein metric with $\lambda > -2$, one can use the $\mathcal{D}$-homothetic deformation to get a Sasakian-Einstein metric.

Let $\rho^T = \mathrm{Ric}^T(\Phi\cdot, \cdot)$ and $\rho = \mathrm{Ric}(\Phi\cdot, \cdot)$. ρ^T is called the *transverse Ricci form*. In view of (3.4) we have

$$\rho^T = \rho + 2d\eta. \tag{3.7}$$

It is easy to see that ρ^T is a closed basic form and its basic cohomology class $[\rho^T]_B = c_B^1$ is the basic first Chern class. c_B^1 is called *positive* (*negative, null* resp.) if it contains a positive (negative, null resp.) representation. The transverse Einstein equation (3.5) can be written as

$$\rho^T = cd\eta, \tag{3.8}$$

for some constant c. A necessary condition for the existence of (3.8) is

$$c_B^1 = c[d\eta]_B.$$

By a $\mathcal{D}$-homothetic deformation, it is natural to consider

$$c_B^1 = \kappa[d\eta]_B, \tag{3.9}$$

where $\kappa = 1, -1, 0$ corresponds to positive, negative and null c_B^1.

Now we consider the following flow $(\xi, \eta(t), \Phi(t), g(t))$ with an initial data $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$

$$\frac{d}{dt}g^T(t) = -(Ric^T_{g(t)} - \kappa g^T(t)), \tag{3.10}$$

or equivalently

$$\frac{d}{dt}d\eta(t) = -(\rho^T_{g(t)} - \kappa d\eta(t)). \tag{3.11}$$

We emphasize that(3.6) and (3.10) are not directly equivalent, since the flow (3.10) is defined in the space of the Sasaki metrics, which is smaller that the space of transverse metrics. (3.6) and (3.10) are equivalent only if we show the short-time existence of (3.10). We call (3.10) *Sasakian-Ricci flow.* In Section 5, we will show that the Sasaki-Ricci flow (3.10) is well-posed.

4. Local computations

In this section, we first review local coordinates on a Sasakian manifold. One can choose local coordinates $(x, z^1, z^2, \cdots, z^n)$ on a small neighborhood U such that

- $\xi = \frac{\partial}{\partial x}$,
- $\eta = dx + \sqrt{-1}\sum_{j=1}^n h_j dz^j - \sqrt{-1}\sum_{j=1}^n h_{\bar{j}} d\bar{z}^j$,
- $\Phi = \sqrt{-1}\{\sum_{j=1}^n\{(\frac{\partial}{\partial z^j} - \sqrt{-1}h_j\frac{\partial}{\partial x}\}\otimes dz^j - \sum_{j=1}^n\{(\frac{\partial}{\partial \bar{z}^j} + \sqrt{-1}h_{\bar{j}}\frac{\partial}{\partial x}\}\otimes d\bar{z}^j\}$,
- $g = \eta\otimes\eta + 2\sum_{j=1}^n h_{j\bar{l}} dz^j d\bar{z}^l$,

where $h : U \to \mathbb{R}$ is a (local) basic function, i.e. $\frac{\partial}{\partial x}h = 0$ and $h_j = \frac{\partial}{\partial z^j}h$ and $h_{j\bar{l}} = \frac{\partial^2}{\partial z^j\partial\bar{z}^l}h$. See for instance [16].

In such local coordinates, $\mathcal{D}^{\mathbb{C}}$ is spanned by

$$X_j := \frac{\partial}{\partial z^j} - \sqrt{-1}h_j\frac{\partial}{\partial x} \text{ and } X_{\bar{j}} := \frac{\partial}{\partial \bar{z}^j} + \sqrt{-1}h_{\bar{j}}\frac{\partial}{\partial x}.$$

Since h is basic, it is clear that

$$\Phi X_j = \sqrt{-1}X_j \quad \text{and} \quad \Phi X_{\bar{j}} = -\sqrt{-1}X_{\bar{j}},$$

$$[X_j, X_l] = [X_{\bar{j}}, X_{\bar{l}}] = [\xi, X_j] = [\xi, X_{\bar{j}}] = 0,$$

and

$$[X_j, X_{\bar{l}}] = 2\sqrt{-1}h_{j\bar{l}}\frac{\partial}{\partial x}.$$

Obviously, $\{\eta, dz^j, dz^{\bar{l}}\}$ is the dual basis of $\{\frac{\partial}{\partial x}, X_i, X_{\bar{j}}\}$. It is clear that the transverse metric

$$g^T = \sum_{j,l=1}^n g^T_{j\bar{l}} dz^j dz^{\bar{l}} = 2\sum_{j=1,l}^n h_{j\bar{l}} dz^j dz^{\bar{l}},$$

and equivalently,

$$d\eta = -2\sqrt{-1}\sum_{j,l=1}^n h_{j\bar{l}} dz^j \wedge dz^{\bar{l}},$$

where $g^T_{j\bar{l}} = g^T(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{l}}}) = 2h_{j\bar{l}}$. Thus we have a simple fact that $\xi g^T_{i\bar{j}} = 0$. Note that

$$g^T(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{l}}}) = g^T(X_j, X_{\bar{l}}). \tag{4.1}$$

From above, we have $\nabla^T_{\frac{\partial}{\partial x}} X_j = \nabla^T_{\frac{\partial}{\partial x}} X_{\bar{l}} = 0$. Define Γ^C_{AB} by

$$\nabla^T_{X_A} X_B = \Gamma^C_{AB} X_C,$$

for $A, B, C = 1, 2, \cdots, n, \bar{1}, \bar{2}, \cdots, \bar{n}$. Since ∇^T is the transverse Levi-Civita connection, i.e., ∇^T satisfies (3.2) and (3.3), one can easily show that only Γ^k_{jl} and $\Gamma^{\bar{k}}_{\bar{j}\bar{l}}$ may not vanish as in the Kähler case. Moreover,

$$\Gamma^k_{jl} = (g^T)^{k\bar{m}} \frac{\partial g^T_{l\bar{m}}}{\partial z^j} = \Gamma^k_{lj},$$

where $((g^T)^{k\bar{m}})$ is the inverse matrix of $(g^T_{j\bar{l}})$.

Note that for a fixed point $p \in M$, one can choose local coordinates $(x, z^1, z^2, \cdots, z^n)$ around p satisfying additionally that

$$\{\frac{\partial}{\partial z^j}\}_p \in \mathcal{D}^{\mathbb{C}}_p,$$

or equivalently $h_j(p) = 0$ for all j. This also means that h attains a local minimum at p. Hence, around point p, one can choose normal coordinates $(x, z^1, z^2, \cdots, z^n)$ such that $\Gamma^k_{jl} = \Gamma^{\bar{k}}_{\bar{j}\bar{l}} = 0$.

One can check that the transverse Ricci curvature can be expressed by

$$R^T_{j\bar{l}} = -\frac{\partial^2}{\partial z^j \partial z^{\bar{l}}} \log \det(g^T_{m\bar{n}}) \tag{4.2}$$

and $\rho^T = \sqrt{-1} R^T_{j\bar{l}} dz^j \wedge dz^{\bar{l}}$.

Now let φ be a smooth basic function, ie., $\xi\varphi = 0$. We consider the new Sasakian structure $(M, \xi, \tilde{\eta}, \widetilde{\Phi}, \tilde{g})$ given by Lemma 3.1 in a local frame. It is easy to compute, for example,

$$\tilde{g}^T = \sum_{j,l=1}^n (g^T_{j\bar{l}} + \varphi_{j\bar{l}}) dz^j dz^{\bar{l}} = 2 \sum_{j=1,l}^n (h_{j\bar{l}} + \frac{1}{2}\varphi_{j\bar{l}}) dz^j dz^{\bar{l}}.$$

Its other structures $(\tilde{\eta}, \widetilde{\Phi}, \tilde{g})$ can be obtained as above by replacing h with $h + \frac{1}{2}\varphi$. This, in fact, provides another proof of Lemma 3.1, see [16].

Recall that $[\rho^T_g]_B = c^1_B = \kappa[d\eta]_B$. By a Hodge decomposition result given in [10], there is a basic function $F : M \to \mathbb{R}$ such that

$$\rho^T_g - \kappa d\eta = d_B d^c_B F. \tag{4.3}$$

We hope to find a new Sasakian structure $(\tilde{\eta}, \widetilde{\Phi}, \tilde{g})$ with $\tilde{\eta} = \eta + d^c_B \varphi$ satisfying

$$\rho^T_{\tilde{g}} = \kappa d\tilde{\eta}.$$

By (4.3) it is equivalent to

$$\rho^T_{\tilde{g}} - \rho^T_g = \kappa d_B d^c_B \varphi + d_B d^c_B F. \tag{4.4}$$

In view of (4.2), in local coordinates (4.4) can be written as

$$\frac{\det(g^T_{j\bar{l}} + \varphi_{j\bar{l}})}{\det(g^T_{j\bar{l}})} = e^{-\kappa\varphi + F}. \tag{4.5}$$

One may expect that the Sasaki-Ricci flow (3.10) can be written as

$$\frac{d}{dt}\varphi = \log\det(g^T_{i\bar{j}} + \varphi_{i\bar{j}}) - \log(\det g^T_{i\bar{j}}) + \kappa\varphi - F, \tag{4.6}$$

for a basic function φ. We will prove it in the following section.

5. Well-posedness of the Sasakian-Ricci flow

In this section, we will prove the following

Theorem 5.1. *Let M^{2n+1} be a compact manifold. For any Sasakian structure (ξ, η, Φ, g) there is a family of Sasakian structures $(\xi(t), \eta(t), \Phi(t), g(t))$ ($t \in [0, T)$ for some constant $T > 0$) with $\xi(t) = \xi$ satisfying (3.10) and the initial condition $(\eta(0), \Phi(0), g(0)) = (\eta, \Phi, g)$.*

One method to prove the well-posedness of the Sasakian-Ricci flow is to develop a transverse parabolic theory. This can be done by following the methods given in [10]. Here we want to give a proof by using the ordinary parabolic theory. First we define

$$\widetilde{\omega} = d\eta + \frac{1}{2}(\nabla d\varphi(\Phi\cdot, \cdot) - \nabla d\varphi(\cdot, \Phi\cdot)) + \frac{1}{2}(\eta \wedge d(\xi\varphi) \circ \Phi + \eta \wedge d\varphi), \tag{5.1}$$

where $\nabla d\varphi$ is the Hessian of φ defined by

$$\nabla d\varphi(X, Y) = X(d\varphi(Y)) - d\varphi(\nabla_X Y)$$

for $X, Y \in TM$. Here ∇ is the Levi-Civita connection with respect to the metric g. It is clear that $\nabla d\varphi$ is symmetric. Let $\varphi_{,i\bar{j}} = X_{\bar{j}} X_i(\varphi) - d\varphi(\nabla_{X_{\bar{j}}} X_i)$ is the covariant derivative with respect to the metric g. One can check that $\overline{\varphi_{,j\bar{l}}} = \varphi_{,l\bar{j}}$ for any j, l.

Lemma 5.2. *$\widetilde{\omega}$ is a two-form on M and satisfies*

$$\widetilde{\omega} = i(g^T_{j\bar{l}} + \varphi_{,j\bar{l}}) dz^j \wedge dz^{\bar{l}}.$$

Hence,

$$\widetilde{\omega}^n \wedge \eta = i^n \det(g^T_{j\bar{l}} + \varphi_{,j\bar{l}}) dz^1 \wedge dz^{\bar{1}} \cdots \wedge dz^n \wedge dz^{\bar{n}} \wedge dx.$$

Moreover, if φ is a basic function, then $\widetilde{\omega} = d\eta + d_B d^c_B \varphi$.

Proof. It is easy to check that

$$\widetilde{\omega}(X, Y) = -\widetilde{\omega}(Y, X) \quad \text{for any} \quad X, Y.$$

On can also compute

$$\widetilde{\omega}(\xi, X) = \widetilde{\omega}(X_j, X_l) = \widetilde{\omega}(X_{\bar{j}}, X_{\bar{l}}) = 0,$$

$$\widetilde{\omega}(X_j, X_{\bar{l}}) = i(g^T_{j\bar{l}} + \varphi_{,j\bar{l}}).$$

$\square$

Now consider the follow equation

$$\frac{d}{dt}\varphi = \log \frac{\widetilde{\omega}^n \wedge \eta}{(d\eta)^n \wedge \eta} + \xi^2\varphi + \kappa\varphi - F, \tag{5.2}$$

for general function φ, i.e., φ in (5.2) need not to be a basic function. Here $\xi^2\varphi = \xi(\xi\varphi)$. This equation is globally defined. In local coordinates, we have

$$\frac{d}{dt}\varphi = \log\det(g^T_{i\bar{j}} + \varphi_{,i\bar{j}}) - \log(\det g^T_{i\bar{j}}) + \xi^2\varphi + \kappa\varphi - F, \tag{5.3}$$

for a function $\varphi : M \to \mathbb{R}$. It is easy to check that (5.3) is an ordinary parabolic equation when

$$\det(g^T_{i\bar{j}} + \varphi_{,i\bar{j}}) > 0. \tag{5.4}$$

Hence by the parabolic theory, for any initial function φ with (5.4), there exists a positive $T > 0$ and a solution $\varphi : M \times [0,T) \to \mathbb{R}$ of (5.3) with (5.4) for any $t \in [0,T)$.

In the sequel, $f_{A_1A_2...A_k}$ means $X_{A_k}...X_{A_1}f$, for any function f and $A_i = 1, 2, \cdots, n, \bar{1}, \bar{2}, \cdots, \bar{n}$. For the simplicity of notation, from now on we simply use $g_{i\bar{j}}$ to denote $g^T_{i\bar{j}}$.

Lemma 5.3. *Equation (5.3) preserves the property $\xi\varphi = 0$.*

Proof. Set $g(t)_{i\bar{j}} = g_{i\bar{j}} + \varphi_{,i\bar{j}}$ and let $(g(t)^{i\bar{j}})$ be the inverse matrix of $(g(t)_{i\bar{j}})$. We compute the evolution equation for $|\xi\varphi|^2$ locally. Note that $\xi g_{i\bar{j}} = 0$, so we have

$$\xi(\varphi_{,i\bar{j}}) = \xi(\varphi_{i\bar{j}} - ig_{i\bar{j}}\xi\varphi) = (\xi\varphi)_{i\bar{j}} - ig_{i\bar{j}}\xi^2\varphi = (\xi\varphi)_{,i\bar{j}}.$$

Therefore, we have

$$\begin{aligned}
&\frac{d}{dt}(\xi\varphi)^2 = 2\xi\varphi\xi\frac{d}{dt}\varphi \\
= \;& 2\xi\varphi\xi(\log\frac{\det(g_{i\bar{j}} + \varphi_{,i\bar{j}})}{\det(g_{i\bar{j}})}) + \xi^2((\xi\varphi)^2) - 2(\xi^2\varphi)^2 + 2\kappa(\xi\varphi)^2 \\
= \;& 2\xi\varphi g(t)^{i\bar{j}}\xi(\varphi_{,i\bar{j}}) + \xi^2((\xi\varphi)^2) - 2(\xi^2\varphi)^2 + 2\kappa(\xi\varphi)^2 \\
= \;& g(t)^{i\bar{j}}[(\xi\varphi)^2]_{,i\bar{j}} - 2g(t)^{i\bar{j}}(\xi\varphi)_i(\xi\varphi)_{\bar{j}} \\
& + \xi^2((\xi\varphi)^2) - 2(\xi^2\varphi)^2 + 2\kappa(\xi\varphi)^2.
\end{aligned}$$

The maximum principle implies that the flow preserves the property that $\xi\varphi = 0$. ∎

Proof of Theorem 5.1. We consider flow (5.2) with an initial function $\varphi(0) = 1$. Flow (5.2) is a parabolic equation. Hence there is a $T > 0$ and $\varphi(t)$ ($t \in [0,T)$) satisfying (5.2) and (5.4) for any $t \in [0,T)$, where $\widetilde{\omega}$ is defined by (5.1). By Lemma 5.3, $\varphi(t)$ is a basic function for each $t \in [0,T)$. Hence by Lemma 3.1, $(\xi, \tilde{\eta}(t), \widetilde{\Phi}(t), \tilde{g}(t))$ is also a Sasakian structure. It is easy to check that $\tilde{g}(t)$ satisfies (3.10). $\square$

6. Global existence and convergence

In this section we will prove the following

Theorem 6.1. *The Sasaki-Ricci flow has a global existence. Moreover, when the basic first Chern class is negative or null, the Sasakian-Ricci flow converges to a limit, which is an η-Einstein metric.*

The proof follows closely the methods given in [8], which in turn follows the methods given by Yau in his resolution of the Calabi conjecture [31]. As mentioned above, the existence of the η-Einstein metric in the case of the negative or null basic first Chern class was already proved in [10]. See also [1].

We have seen that the Sasaki-Ricci flow (3.10) can be written as

$$\frac{d}{dt}\varphi = \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} + \kappa\varphi - F. \tag{6.1}$$

Recall that in this section we use $g_{i\bar{j}}$ to denote the transverse metric $g^T_{i\bar{j}}$. We now try to get estimates which will imply the global existence. Differentiating the equation (6.1) with respect to t, we then have

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right) = \triangle^t_B\left(\frac{\partial \varphi}{\partial t}\right) + \kappa \frac{\partial \varphi}{\partial t}. \tag{6.2}$$

Here $\triangle^t_B$ is the basic Laplacian with respect to the transverse metric $g^T(t)$. Let $\triangle_t$ denote the ordinary Laplacian with respect to the metric $g(t)$. Acting on basic function, both Laplacian are the same, i.e., $\triangle_t \varphi = \triangle^t_B \varphi$, for a basic function φ. If there is no confusion, we will omit the subscript (or superscript) t.

Let $C(\kappa, T)$ and $c_i(\kappa, T), i \in \mathbb{N}$ be positive constants satisfying

$$C(\kappa, T) + c_i(\kappa, T) \leq \begin{cases} C_0, & \text{if } \kappa \leq 0 \\ C(T), & \text{if } \kappa > 0 \end{cases}$$

where C_0 is a constant depending only on the initial metric which may change from line to line and $C(T)$ is a finite constant depending on T. These constants serve as bounds for the solutions of (6.1). As one will see, in case of $\kappa > 0$, $|\varphi|$ has at most exponential growth rate and $||\varphi||_{C^k}$ may have higher growth rate.

Applying the maximum principle to (6.2), we have

$$\sup_{[0,T)\times M} \left|\frac{\partial \varphi}{\partial t}\right| \leq C(\kappa, T).$$

Let ψ be the normalization of φ such that its mean value is zero with respect to the background metric g, i.e.

$$\psi = \varphi - \int_M \varphi d\mu_g.$$

Proposition 6.2. **(C^0 estimates)** *Assume the Sasaki-Ricci flow (6.1) exists for $t \in [0,T)$, $T < \infty$. Then we have $\sup_{M\times[0,T)} |\psi| \le C(\kappa, T)$.*

Proof. (1) $c_B^1 < 0$. It is clear from the maximum principle. In this case we have the equation $\frac{d}{dt}\varphi = \log \frac{det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{det(g_{i\bar{j}})} - \varphi - F$. Therefore, the statement is easy to check by the maximum principle.

(2) $c_B^1 > 0$. In this case we have the equation $\frac{d}{dt}\varphi = \log \frac{det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{det(g_{i\bar{j}})} + \varphi - F$. It is easy to show that $\sup_{M\times[0,T)} \varphi \le C_0(e^T + 1)$.

(3)$c_B^1 = 0$. We need to consider the equation $\frac{d}{dt}\varphi = \log \frac{det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{det(g_{i\bar{j}})} - F$. For such a equation, we follow Yau's idea in the proof of Calabi-Yau theorem [31] to use the Nash-Moser iteration argument to show the Proposition.

Note that φ is actually not necessarily uniformly bounded in this case. However, it becomes easier if we consider ψ, the normalization of φ.

Lemma 6.3. *There exists a positive constant C_1 such that*

$$\sup_{M\times[0,T)} \psi \le C_1, \qquad \sup_{t\in[0,T)} \int_M |\psi(t)| d\mu_g \le 2C_1 V.$$

The proof is based on the fact $\triangle_B \psi + n > 0$, which follows from the positivity of $g^T(t)$. Let G be the Green function (with respect to $\triangle$) on M satisfying

$$0 \le G(x,y) \le \frac{c}{d(x,y)^{2n-1}}.$$

Note that $\triangle_B \psi = \triangle \psi$, then for any $x \in M$ we have

$$\psi(x) = \frac{-1}{V}\int_M \triangle\psi(y) G(x,y) d\mu_g(y) \le \frac{n}{V}\int_M G(x,y) d\mu_g(y) \le C_1.$$

The second inequality follows from

$$\int_M |\psi| d\mu_g \le \int_M |\psi - 2C_1| d\mu_g = \int_M (2C_1 - \psi) d\mu_g = 2C_1 V.$$

Lemma 6.4. *There exists a constant $C_2 > 0$ such that* $\sup_{M\times[0,T)} |\psi| \le C_2$.

Now we apply the Nash-Moser iteration to achieve this. Let $\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^j$, $\omega_1 = i g'_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Here $g'_{i\bar{j}} = g_{i\bar{j}} + \psi_{i\bar{j}}$. Let $\psi' = C_1 + 1 - \psi \ge 1$.

We have $\omega_1 = \omega - i\partial_B\overline{\partial}_B \psi'$, then we see that

$$(\exp(\frac{\partial\varphi}{\partial t} + F) - 1)\omega^n = \omega_1^n - \omega^n = -i\partial_B\overline{\partial}_B\psi' \wedge (\omega_1^{n-1} + \omega_1^{n-2}\wedge\omega + \dots + \omega^{n-1}),$$

where both ω and ω_1 are metrics, so that $\omega_1^k \wedge \omega^{n-k} \wedge \eta$ defines a volume form. Thus for any $p \ge 1$, we have

$$
\begin{aligned}
& -\frac{1}{V}\int_M i(\psi')^p \partial_B\overline{\partial}_B\psi' \wedge (\omega_1^{n-1} + \omega_1^{n-2}\wedge\omega + \dots + \omega^{n-1})\wedge\eta \\
\geq\ & \frac{1}{V}\int_M i\partial_B(\psi')^p \wedge \overline{\partial}_B\psi' \wedge \omega^{n-1}\wedge\eta \\
=\ & \frac{p}{V}\int_M i(\psi')^{p-1}\partial_B\psi' \wedge \overline{\partial}_B\psi' \wedge \omega^{n-1}\wedge\eta \\
=\ & \frac{4p}{V(p+1)^2}\int_M i\partial_B(\psi'^{\frac{p+1}{2}}) \wedge \overline{\partial}_B(\psi'^{\frac{p+1}{2}}) \wedge \omega^{n-1}\wedge\eta \\
=\ & \frac{4p}{n(p+1)^2}\frac{1}{V}\int_M |\nabla(\psi'^{\frac{p+1}{2}})|^2\omega^n\wedge\eta \\
\geq\ & \frac{4p}{n(p+1)^2}[c_1(\frac{1}{V}\int_M |(\psi')^{\frac{p+1}{2}}|^{\frac{4n+2}{2n-1}}\omega^n\wedge\eta)^{\frac{2n-1}{2n+1}} - \frac{c_2}{V}\int_M |(\psi')^{\frac{p+1}{2}}|^2\omega^n\wedge\eta],
\end{aligned}
$$

the last inequality follows from Sobolev inequality.

On the other hand, we have

$$
\begin{aligned}
& -\frac{1}{V}\int_M i(\psi')^p \partial_B\overline{\partial}_B\psi' \wedge (\omega_1^{n-1} + \omega_1^{n-2}\wedge\omega + \dots + \omega^{n-1})\wedge\eta \\
=\ & \frac{1}{V}\int_M (\psi')^p(\exp(\frac{\partial\varphi}{\partial t} - F) - 1)\omega^n\wedge\eta \\
\leq\ & \frac{c}{V}\int_M (\psi')^p\omega^n\wedge\eta \\
\leq\ & \frac{c}{V}\int_M (\psi')^{p+1}\omega^n\wedge\eta.
\end{aligned}
$$

Putting all these together, we then have

$$
c_1(\frac{1}{V}\int_M |(\psi')^{\frac{p+1}{2}}|^{\frac{4n+2}{2n-1}}d\mu_g)^{\frac{2n-1}{2n+1}} - \frac{c_2}{V}\int_M |(\psi')^{\frac{p+1}{2}}|^2 d\mu_g \leq \frac{cp}{V}\int_M (\psi')^{p+1}d\mu_g,
$$

which implies

$$
(\frac{1}{V}\int_M |(\psi')^{(p+1)\frac{2n+1}{2n-1}}d\mu_g)^{\frac{2n-1}{2n+1}} \leq \frac{C(p+1)}{V}\int_M (\psi')^{p+1}d\mu_g.
$$

Therefore, we have $\|\psi'\|_{L^{(p+1)\frac{2n+1}{2n-1}}} \leq (C(p+1))^{\frac{1}{p+1}}\|\psi'\|_{L^{p+1}}$.

Choose $p_0 = 1$ and define p_k by $p_k + 1 = \frac{2n+1}{2n-1}(p_{k-1}+1)$, then for $i \geq 1$, we have

$$
\|\psi'\|_{L^{p_i+1}} \leq \prod_{k=0}^{i-1}[C(p_k+1)]^{\frac{1}{p_k+1}}\|\psi'\|_{L^2}.
$$

Now if we let $i \to \infty$, then $p_i + 1 \to \infty$, so

$$
\sup_M \psi' = \lim_{i\to\infty}\|\psi'\|_{L^{p_i+1}} \leq \prod_{k=0}^{\infty}[C(p_k+1)]^{\frac{1}{p_k+1}}\|\psi'\|_{L^2}.
$$

So we have shown that $\sup_M |\psi'| = \sup_M \psi' \leq C\|\psi'\|_{L^2}$.

Furthermore, we have

$$\begin{aligned} & \frac{c}{V}\int_M \psi'\omega^n \wedge \eta \\ \geq\ & \frac{1}{V}\int_M \psi'(\exp(\frac{\partial\varphi}{\partial t} + F) - 1)\omega^n \wedge \eta \\ =\ & \frac{1}{V}\int_M \psi'((\omega - i\partial\overline{\partial}\psi')^n - \omega^n) \wedge \eta \\ \geq\ & \frac{1}{nV}\int_M |\nabla\psi'|^2\omega^n \wedge \eta \\ \geq\ & \frac{\lambda_1(\omega)}{nV}[\int_M |\psi'|^2\omega^n \wedge \eta - (\int_M \psi'\omega^n \wedge \eta)^2], \end{aligned}$$

where the last inequality follows from the Poincarè inequality. So we deduce from lemma 6.3 that

$$\|\psi'\|_{L^2} \leq c(\|\psi'\|_{L^1} + 1) \leq C.$$

Hence $\sup_M |\psi'(t)| \leq C$, thus $\sup_M |\psi(t)| \leq C$. □

Proposition 6.5. **(C^2 estimates)** *Assume the Sasaki-Ricci flow (3.10) (or (6.1)) exists for $t \in [0, T)$, $T < \infty$. Then we have $\sup_{t\in[0,T)} |\varphi|_{C^2} \leq C(\kappa, T)$.*

Proof. First one can compute (cf. [27])

$$\begin{aligned} & (\triangle'_B - \partial_t)\log(n + \triangle_B\varphi) \\ =\ & \frac{1}{n + \triangle_B\varphi} g^{i\bar{j}}g'^{k\bar{t}}g'^{s\bar{l}}\nabla_i\varphi_{s\bar{t}}\nabla_{\bar{j}}\varphi_{k\bar{l}} - \frac{|\nabla'_B\triangle_B\varphi|^2}{(n + \triangle_B\varphi)^2} \\ & + \frac{1}{n + \triangle_B\varphi}(g'^{k\bar{l}}g'_{i\bar{j}}\ {}^T R^{i\bar{j}}{}_{k\bar{l}} - R^T - \kappa\triangle_B\varphi + \triangle_B F) \end{aligned}$$

and

$$g^{i\bar{j}}g'^{k\bar{t}}g'^{s\bar{l}}\nabla_i\varphi_{s\bar{t}}\nabla_{\bar{j}}\varphi_{k\bar{l}} \geq \frac{|\nabla'_B\triangle_B\varphi|^2}{n + \triangle_B\varphi}.$$

In normal coordinates centered in a fixed point p (See section 4) we have

$$\begin{aligned} (\triangle'_B - \partial_t)\log(n + \triangle_B\varphi) &\geq \frac{1}{n + \triangle_B\varphi}(g'^{k\bar{l}}g'_{i\bar{j}}\ {}^T R^{i\bar{j}}{}_{k\bar{l}} - \kappa\triangle_B\varphi - c) \\ &\geq -\frac{c}{n + \triangle_B\varphi}(\sum_{i,k}\frac{1 + \varphi_{i\bar{i}}}{1 + \varphi_{k\bar{k}}} + 1) - \kappa \\ &\geq -c_0\sum_k \frac{1}{1 + \varphi_{k\bar{k}}} - \kappa. \end{aligned}$$

On the other hand, we have

$$(\triangle'_B - \partial_t)\psi = \sum_k \frac{\varphi_{k\bar{k}}}{1 + \varphi_{k\bar{k}}} - \frac{\partial\psi}{\partial t} = -\sum_k \frac{1}{1 + \varphi_{k\bar{k}}} + n - \frac{\partial\psi}{\partial t}.$$

Note that $\frac{\partial\psi}{\partial t} = \frac{\partial\varphi}{\partial t} + c(t)$ and $\int_M \frac{\partial\psi}{\partial t} d\mu_g = 0$, thus $\sup_M |\frac{\partial\psi}{\partial t}| \le 2\sup_M |\frac{\partial\varphi}{\partial t}|$. Therefore we have

$$(\triangle'_B - \partial_t)[\log(n + \triangle_B\varphi) - (c_0+1)\psi] \ge \sum_k \frac{1}{1+\varphi_{k\bar{k}}} - c_1(\kappa, T) \tag{6.3}$$

at the fixed point p. For any given $t \in (0, T)$, assume that $u := \log(n + \triangle_B\varphi) - (c_0+1)\psi$ achieves its maximum at point $(p_0, t_0) \in M \times (0, t]$. We consider normal coordinates centered at p_0. Since (t_0, p_0) is a maximum point, $(\triangle'_B - \partial_t)u \le 0$, and hence from (6.3) we have $\sum_k \frac{1}{1+\varphi_{k\bar{k}}} \le c_1(\kappa, T)$ at this point. On the other hand, we have

$$(\sum_k \frac{1}{1+\varphi_{k\bar{k}}})^{n-1} \ge (n + \triangle_B\varphi)(\prod_k \frac{1}{1+\varphi_{k\bar{k}}}) = (n + \triangle_B\varphi)e^{-\frac{\partial\varphi}{\partial t} + \kappa\varphi - F}.$$

Thus at the point (p_0, t_0), we have $n + \triangle_B\varphi \le c_2(\kappa, T)$. By the definition of u, we see

$$\sup_M (n + \triangle_B\varphi) \le c_3(\kappa, T).$$

Furthermore, since

$$\sum_k \frac{1}{1+\varphi_{k\bar{k}}} \le (n + \triangle_B\varphi)^{n-1} \prod_k \frac{1}{1+\varphi_{k\bar{k}}} = (n + \triangle_B\varphi)^{n-1} e^{-\frac{\partial\varphi}{\partial t} + \kappa\varphi - F},$$

we have

$$\sup_M \sum_k \frac{1}{1+\varphi_{k\bar{k}}} \le c_4(\kappa, T).$$

Note that

$$\prod_k (1 + \varphi_{k\bar{k}}) = e^{\frac{\partial\varphi}{\partial t} - \kappa\varphi + F}.$$

Therefore there exists $c_6(\kappa, T) \ge c_5(\kappa, T) > 0$ such that

$$c_5(\kappa, T) \le 1 + \varphi_{k\bar{k}} \le c_6(\kappa, T).$$

This also promises that g' stays as metrics. □

Proposition 6.6. **(C^3 estimates)** *Assume the Sasaki-Ricci flow (3.10) (or (6.1)) exists for $t \in [0, T)$, $T < \infty$. Then we have $\sup_{t\in[0,T)} |\varphi|_{C^3} \le C(\kappa, T)$.*

Proof. Due to Calabi and Yau, one can consider the quantity

$$S = g^{i\bar{r}} g^{s\bar{j}} g^{k\bar{t}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}}.$$

Similar calculation as in [31] shows that

$$(\triangle'_B - \partial_t)(S + c_7(\kappa, T)\triangle_B\varphi) \ge c_8(\kappa, T)S - c_9(\kappa, T).$$

At the maximum point $p(t)$ of $(S + c_7(\kappa, T)\triangle_B\varphi)$ at time t, we have $c_8(\kappa, T)S - c_9(\kappa, T) \le 0$. On the other hand $\triangle_B\varphi \le C(\kappa, T)$, hence $\sup_{M\times[0,T)} S$ is bounded, i.e. $\sup_{t\in[0,T)} |\varphi|_{C^3} \le C(\kappa, T)$. □

Proof of Theorem 6.1. With these estimates, one see the flow exists for all time (cf. [8]). More precisely, if we suppose the Sasaki-Ricci flow exists for a maximal time interval $[0, T)$, one can see T cannot be finite. Otherwise, $\sup_{t\in[0,T)} |\varphi|_{C^3} \leq C(\kappa, T)$ implies that $g^T(T)$ is a smooth metric. Therefore the local existence of the Sasaki-Ricci flow tells that the flow continues after time T, which is a contradiction.

In the case c_B^1 is null, the convergence of the Sasaki Ricci flow follows from Li-Yau's Harnack inequality (cf. [8]).

Because of equation

$$\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t}\right) = \triangle_B^t\left(\frac{\partial\varphi}{\partial t}\right) + \kappa\frac{\partial\varphi}{\partial t},$$

the maximum principle implies the exponential decay of $|\frac{\partial\varphi}{\partial t}|$ in case of c_B^1 is negative. So we conclude that as $t \to \infty$, $g(t)$ converges in such case.

We now have finished the proof of Theorem 6.1. $\square$

A similar flow as in [8] can also be considered to prove the transverse Calabi-Yau theorem [10]. Let (M, η, ξ, Φ, g) be a Sasakian manifold with transversal Kähler form ω and transversal Ricci form ρ_η^T, denoted by

$$\omega = d\eta = ig_{j\bar{l}}dz^j \wedge d\bar{z}^l, \quad \rho_\eta^T = i\,R_{j\bar{l}}^T dz^j \wedge d\bar{z}^l.$$

Let $T = iT_{j\bar{l}}dz^j \wedge d\bar{z}^l \in \Lambda_B^{1,1}$ and $[T] = c_B^1$. Therefore, there exist a real basic function F such that

$$\rho_\eta^T - T = i\partial_B\overline{\partial}_B F, \quad i.e. \quad R_{j\bar{l}}^T - T_{j\bar{l}} = F_{j\bar{l}}.$$

Let $\widetilde{g}_{j\bar{l}} = g_{j\bar{l}} + \varphi_{j\bar{l}}$, where φ is a real basic function. Then we have

$$\widetilde{R}_{j\bar{l}}^T = -\partial_j\partial_{\bar{l}} \log det(g_{r\bar{s}} + \varphi_{r\bar{s}}).$$

The problem of finding $\widetilde{\omega} \in [d\eta]_B$ which has T as its transversal Ricci form is to solve the equation

$$\log\frac{det(g_{j\bar{l}} + \varphi_{j\bar{l}})}{det(g_{j\bar{l}})} = F.$$

The flow equation can be defined by

$$\frac{\partial\varphi}{\partial t} = \log\frac{det(g_{j\bar{l}} + \varphi_{j\bar{l}})}{det(g_{j\bar{l}})} - F, \tag{6.4}$$

i.e.

$$\frac{\partial}{\partial t}\widetilde{g}_{j\bar{l}} = T_{j\bar{l}} - \widetilde{R}_{j\bar{l}}^T.$$

One can check that equation (6.4) is well defined and preserves the basic condition for φ. Note that the flow shares the same equation as the Sasaki Ricci flow of the case $c_B^1 = 0$. In the same way, one has same estimates for equation (6.4). This implies the long time existence and convergence.

7. Further questions

As in the Kähler case, the Sasakia-Ricci flow in the positive case becomes more difficult and intersting. In general one could not expect that the flow will converge to a transverse Einstein metric (equivalently η-Einstein metric). One may expect that the flow converges to a soliton type solution. Motivated by the Sasaki-Ricci flow considered in this paper, Futaki, One and the second author introduced the Sasaki-Ricci soliton in [13] and consider the existence of Sasaki-Ricci solitons on Sasaki toric manifolds and proved the existence of Sasaki-Einstein metric on any Sasaki toric manifolds. In general, we expect that the Sasaki-Ricci flow converges in a suitable sense to a Sasaki-Ricci soliton. In the Kähler case such a result was proved by Tian and Zhu [29] inspired by Perelman's work in the Ricci flow [24] and [25]. See also [26] In the another direction, the transverse differential geometry helps us to give a natural generalization of Futaki invariant [11] and [12]. In [13] we introduced a Sasaki-Futaki invariant, which is an obstruction of the existence of Sasaki-Einstein metric. A similar invariant was independently introduced by Boyer, Galicki and Simanca [6]. In that paper they also studied Sasaki extremal metrics and consider such a metric as a canonical metric on Sasaki manifold.

References

[1] Boyer, C. P.; Galicki, K. On Sasakian-Einstein geometry. Internat. J. Math. **11** (2000) 873–909.

[2] Boyer, C. P.; Galicki, K., New Einstein metrics in dimension five, *J. Differential Geom.* **57** (2001) 443–463.

[3] Boyer, C. P.; Galicki, K.; Kollór, J., Einstein metrics on spheres, *Ann. of Math.,* (2) **162** (2005)557–580.

[4] C.P. Boyer, K. Galicki, and P. Matzeu, *On Eta-Einstein Sasakian Geometry*, Comm. Math. Phys., 262 (2006), pp. 177-208.

[5] Boyer, C. P.; Galicki, K.; Nakamaye, M., On the geometry of Sasakian-Einstein 5-manifolds, *Math. Ann.,* **325** (2003) 485–524.

[6] C.P. Boyer, K. Galicki, and R. Simanca, Canonical Sasakian Metrics, **Arxiv:math.DG/0604325**

[7] Chen, X. X.; Tian, G., Ricci flow on Khler-Einstein surfaces. Invent. Math. 147 (2002), no. 3, 487–544.

[8] Cao, Huai Dong, Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds,*Invent. Math.,* **81** (1985) 359–372.

[9] M. Cvetic, H. Lü, Don N. Page, and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, Phys. Rev. Lett., **95** (2005)

[10] El Kacimi-Alaoui, A. *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications,* Compositio Math. 79 (1990), 57–106.

[11] A. Futaki. An obstruction to the existence of Einstein Kähler metrics. *Invent. Math.*, **73** (1983) 437–443.

[12] A. Futaki. Kähler-Einstein metrics and integral invariants. *Lecture Notes in Mathematics,* **1314**, Springer-Verlag, Berlin, 1988.

[13] A. Futaki, H. Ono and G. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds **Arxiv:math.DG/0607586**

[14] J.P. Gauntlett, D. Martelli, J. Sparks and W. Waldram, Sasaki-Einstein metrics on $S^2 \times S^3$, *Adv. Theor. Math. Phys.,* **8** (2004) 711-734

[15] J.P. Gauntlett, D. Martelli, J. Sparks and W. Waldram, *A new infinite class of Sasaki-Einstein manifolds*, 2004, Adv. Theor. Math. Phys., **8** (2004), 987-1000

[16] M. Godliński, W. Kopczyński and P. Nurowski, Locally Sasakian manifolds. *Classical Quantum Gravity,* **17** (2000) L105–L115.

[17] Hamilton, Richard S. Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255–306.

[18] Lovrić, Miroslav; Min-Oo, Maung; Ruh, Ernst A. Deforming transverse Riemannian metrics of foliations. Asian J. Math. 4 (2000), no. 2, 303–314.

[19] H. Lü, C.N. Pope, J.F. Vazquez-Poritz, A New Construction of Einstein-Sasaki Metrics in $D \geq 7$, **Arxiv:hep-th/0512306**

[20] D. Martelli and J. Sparks, *Toric Sasaki-Einstein metrics on* $S^2 \times S^3$, Phys. Lett. B, **621** (2005), 208-212

[21] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, Comm. Math. Phys., **262** (2006) 51-89.

[22] D. Martelli, J. Sparks and S.-T. Yau, *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*, **arXiv:hep-th/0503183**

[23] D. Martelli, J. Sparks and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, **arXiv:hep-th/0603021**

[24] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, **Arxiv: math.DG/0211159**

[25] G.Perelman, Ricci flow with surgery on three-manifolds, **Arxiv:math.DG/0303109**

[26] N. Sesum and G. Tian, Bounding scalar curvature and diameter along the Kaehler-Ricci flow (after Perelman) and some applications. Preprint

[27] Siu, Yum Tong,Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics. DMV Seminar, 8. Birkhäuser Verlag, Basel, 1987.

[28] G. Tian, Canonical metrics in Kähler geometry. Notes taken by Meike Akveld. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000.

[29] G. Tian and X. Zhu, Convergence of Kähler-Ricci flow, to appear in J.A.M.S.

[30] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds,* Progress in Mathematics, 30. Birkhäuser, Boston, Mass., 1983.

[31] Yau, Shing Tung, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.

Leibniz Universität Hannover, Institut für Differentialgeometrie, Welfengarten 1, 30167 Hannover, Germany

E-mail address: smoczyk@math.uni-hannover.de

University of Magdeburg, Faculty of Mathematics, P.O.Box 4120 , D-39016 Magdeburg, Germany

E-mail address: guofang.wang@math.uni-magdeburg.de

Leibniz Universität Hannover, Institut für Differentialgeometrie, Welfengarten 1, 30167 Hannover, Germany

E-mail address: ybzhang@amss.ac.cn