ON ALGEBRAIC SELFSIMILAR SOLUTIONS OF THE MEAN CURVATURE FLOW

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Abstract. In this short note we show that in codimension one all homogeneous algebraic selfsimilar solutions of the mean curvature flow are either algebraic minimal hypersurfaces or belong to the class of well known self-shrinking quadrics.

1. Selfsimilar solutions of the mean curvature flow

Let M be a smooth manifold of dimension n and $f_0 : M \to \mathbb{R}^{n+k}$, $k \geq 1$, be a smooth immersion. A smooth map $f : M \times [0, T) \to \mathbb{R}^{n+k}$, $T > 0$, is called a solution of the mean curvature flow with initial condition f_0, if each map $f_t := f(\cdot, t) : M \to \mathbb{R}^{n+k}$ is an immersion and f satisfies

$$\frac{d}{dt} f(x,t) = \overrightarrow{H}(x,t)\,, \quad f(x,0) = f_0(x)\,, \forall\, x \in M\,, \tag{1}$$

where $\overrightarrow{H}(x,t)$ is the mean curvature vector of the immersion f_t. There is an important class of solutions to (1), namely the selfsimilar solutions. These are immersions $f : M \to \mathbb{R}^{n+k}$ that satisfy the elliptic system

$$\overrightarrow{H}(x) = -\mu f^{\perp}(x) \tag{2}$$

with some constant $\mu \in \mathbb{R}$. Here $f^{\perp}$ denotes the normal part of the position vector f. A solution of (2) is called shrinking, stationary (minimal) or expanding depending on whether $\mu > 0$, $\mu = 0$ or $\mu < 0$. Under mean curvature flow, selfsimilar solutions just evolve by homotheties (or are stationary, if $\mu = 0$) and it has been shown in [Hui93] that they play an important role in the formation of singularities of this flow. For curves there was a complete classification of all solutions to (2) given by Abresch and Langer [AL86]. Huisken [Hui93] classified all solutions M, where M is a mean convex hypersurface in $\mathbb{R}^{n+1}$. In particular, for $n > 1$ all closed hypersurfaces that solve (2) and are mean convex (or star-shaped) are round spheres. Moreover, any minimal submanifold of the sphere S^{n+k-1} is also a self-shrinker in $\mathbb{R}^{n+k}$. In [Smo05] the author proved that a self-shrinker in arbitrary codimension is a minimal submanifold of the sphere, if and only if the mean curvature vector $\overrightarrow{H}$ is nonvanishing and its principal normal $\nu = \overrightarrow{H}/|\overrightarrow{H}|$ is

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parallel. There are a few more results for special types of mean curvature flows, as for example for the Lagrangian mean curvature flow ([JLT08], [CL09]).

Selfsimilar solutions of the mean curvature flow behave in many ways like minimal submanifolds, in fact since they are solutions of the Euler-Lagrange equation for the functional

$$E(f) = \int_M e^{-\frac{\mu |f|^2}{2}} \, \mathrm{dvol}_M$$

they are indeed minimal submanifolds in $\mathbb{R}^{n+k}$ equipped with the metric

$$\bar{g}(y)(V,W) := e^{-\frac{\mu |y|^2}{n}} \langle V, W \rangle$$

which is conformally equivalent to the euclidean metric.

From this observation it is clear that the theory of selfsimilar solutions is a subbranch of minimal surface theory (e.g. see [CM09] and [Smo01]) and all questions and concepts that arise for minimal submanifolds are equally important for the class of selfsimilar solutions. Unfortunately there are only a few explicit examples for selfsimilar solutions, in particular for selfsimilar hypersurfaces. Besides the standard examples of Abresch-Langer curves [AL86], spheres $S^n(r = \sqrt{n/\mu})$, generalized cylinders $S^l(r = \sqrt{l/\mu}) \times \mathbb{R}^{n-l} \subset \mathbb{R}^{n+1}$ and those tori $f : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}$ found by Angenent [Ang92] there are almost no explicit examples. For example, it is unknown, if the standard spheres are the only embedded selfsimilar spheres in $\mathbb{R}^{n+1}$. However, it is known that the standard spheres are the only embedded stable selfsimilar spheres.

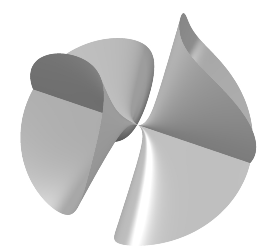

FIGURE 1. The Henneberg minimal surface is an algebraic minimal surface.

One attempt to find more explicit examples is to construct them as real algebraic solutions. The situation is already very interesting in the stationary case, i.e. when M is minimal and $\mu = 0$. Minimal algebraic submanifolds exist. One example is the quadric 3-fold $M = \{y \in \mathbb{R}^4 : (y^1)^2 + (y^2)^2 - (y^3)^2 - (y^4)^2 = 0\}$, where $y = (y^1, y^2, y^3, y^4) \in \mathbb{R}^4$. This is a homogeneous algebraic minimal hypersurface, more precisely the cone generated by the Clifford torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3 \subset \mathbb{R}^4$. The Henneberg surface [Hen78] (Figure 1) is an algebraic minimal surface of degree 15.

There exist other algebraic minimal surfaces, for example the famous Enneper surface is a nonic minimal surface (Figure 2).

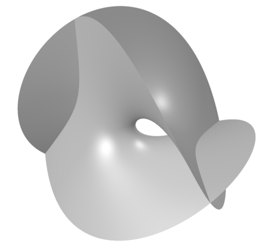

FIGURE 2. The Enneper surface is a nonic minimal surface given by the equation:
$(x^2 - y^2 + \frac{4}{3}z + \frac{4}{9}z^3)^3 - 3z(x^2 - y^2 + \frac{8}{9}z - z(x^2 + y^2 + \frac{8}{9}z^2))^2 = 0.$

To our knowledge it is not known, if non-trivial complete algebraic minimal surfaces in $\mathbb{R}^3$ can be embedded. In fact, until the discovery of Costa [Cos84], Hoffman and Meeks [HM85] it was widely conjectured that besides the plane, the catenoid and the helicoid there are no other complete embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type. Since then many other embedded minimal surfaces have been found.

Recall that, by the classical results of Osserman [Oss63] and Chern-Osserman [CO67], a complete connected minimal surface $M^2 \subset \mathbb{R}^m$ with finite total curvature is conformally equivalent to a punctured Riemann surface. Therefore one sometimes also calls these minimal surfaces algebraic. That is not what we do in this paper. For us "algebraic" means the level set of a real polynomial.

In higher codimension there exist some interesting examples of algebraic selfsimilar solutions. Any minimal surface of the sphere is a self-shrinker and the intersection of S^n with an algebraic minimal cone in $\mathbb{R}^{n+1}$ gives an algebraic selfsimilar solution of codimension two. In [Per07], [Per05] Perdomo dealt with order d-algebraic minimal hypersurfaces $M \subset S^n$, i.e. with minimal immersions $M \subset S^n$ with the property that for some irreducible homogeneous polynomial $f : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree d, $M = f^{-1}(0) \cap S^n$. Some known examples of algebraic minimal immersions in the sphere are the isoparametric hypersurfaces of the sphere. The order of these examples can only be 1, 2, 3, 4 and 6. Another family of algebraic examples were found by Lawson in [Law70], where he proved that the polynomial $f(x) = \mathrm{Im}\{(y^1 + iy^2)^m (y^3 - iy^4)^l\}$ defines an algebraic surface of order $d = l + m$ in S^3. Some additional examples were found by Hsiang in [Hsi67].

In this article we restrict our attention to homogeneous algebraic selfsimilar hypersurfaces. According to Remark 3.2b) they form the most natural class to consider. Our result then states that in this class we do not find any new examples. All homogeneous algebraic selfsimilar hypersurfaces are either minimal or one of those self-shrinking quadrics mentioned above. More precisely we prove:

Theorem 1.1. *Suppose M is an irreducible homogeneous algebraic selfsimilar hypersurface of the mean curvature flow in $\mathbb{R}^{n+1}$. Then one of the following cases holds true:*

1.) M is regular and either

i) M is a regular homogeneous algebraic minimal hypersurface, or
ii) M is of the type $M = S^l(r) \times \mathbb{R}^{n-l}$ with $1 \leq l \leq n$. In particular, M is a self-shrinker of degree 2, $\mu > 0$, $r = \sqrt{l/\mu}$.

2.) M is singular and M is a minimal projective variety generated by a homogeneous algebraic minimal hypersurface $\tilde{\Gamma}$ of the standard sphere S^n.

Moreover, in codimension one there do not exist any homogeneous algebraic self expanders of the mean curvature flow.

As mentioned above, it is unclear if - besides the plane - there are any other regular homogeneous algebraic minimal hypersurfaces.

2. Hypersurfaces as level sets

Suppose $p : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function. A point $y \in \mathbb{R}^{n+1}$ is called a regular point of p, if $Dp(y) \neq 0$. Let

$$M_a := \{y \in \mathbb{R}^{n+1} : p(y) = a\}$$

be the a-level set of p. The value a is called a regular value of p, if all $y \in M_a$ are regular points of p. Throughout the paper let us assume $M_a \neq \emptyset$. A set M_a is called regular w.r.t. p, if M_a is the a-level set of p and a is a regular value of p. In the sequel "M_a being regular" always means "regular w.r.t. a (fixed) function p". If $y \in \mathbb{R}^{n+1}$ is a regular point with $p(y) = a$, then by the implicit function theorem, there exists a small open neighborhood $U \subset \mathbb{R}^{n+1}$ around y such that $M := M_a \cap U$ is a smooth embedded hypersurface and such that all $y \in M$ are regular points. Suppose $f : \Omega \to M$ is a smooth embedding. Then

$$(p \circ f)(x) = a\,, \quad \forall x \in \Omega \tag{3}$$

It is then standard to obtain local formulas for the first and second fundamental tensors. Taking a partial derivative of (3) in local coordinates $(x^i)_{i=1,\dots,n}$ and Cartesian coordinates $(y^\alpha)_{\alpha=1,\dots,n+1}$ on $\mathbb{R}^{n+1}$ one gets

$$0 = \frac{\partial p}{\partial y^\alpha}(f(x))\frac{\partial f^\alpha}{\partial x^i}(x) = \langle Dp(f(x)), f_i(x)\rangle\,, \tag{4}$$

with

$$Dp(y) = \delta^{\alpha\beta}\frac{\partial p}{\partial y^\alpha}(y)\frac{\partial}{\partial y^\beta}$$

and

$$f_i(x) = \frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial}{\partial y^\alpha}\,.$$

Since $y = f(x) \in M$ is a regular point (4) implies that a unit normal $\nu(x)$ at $f(x)$ is given by

$$\nu(x) = -\frac{Dp(f(x))}{||Dp(f(x))||}\,, \tag{5}$$

where $||\cdot||$ denotes the euclidean norm in $\mathbb{R}^{n+1}$. Taking another derivative of (4) one obtains

$$\begin{aligned}
0 &= \frac{\partial^2 p}{\partial y^\alpha \partial y^\beta}(f(x))\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\beta}{\partial x^j}(x) + \frac{\partial p}{\partial y^\beta}(f(x))\frac{\partial^2 f^\beta}{\partial x^i \partial x^j}(x)\\
&\overset{(4)}{=} \frac{\partial^2 p}{\partial y^\alpha \partial y^\beta}(f(x))\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\beta}{\partial x^j}(x) + \frac{\partial p}{\partial y^\beta}(f(x))\Big(\frac{\partial^2 f^\beta}{\partial x^i \partial x^j}(x) - \Gamma^k_{ij}(x)\frac{\partial f^\beta}{\partial x^k}(x)\Big)\\
&= \frac{\partial^2 p}{\partial y^\alpha \partial y^\beta}(f(x))\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\beta}{\partial x^j}(x) + \frac{\partial p}{\partial y^\beta}(f(x))A^\beta_{ij}(x)\,,
\end{aligned}$$

where $\Gamma^k_{ij}(x)$ is the Christoffel symbol of the induced metric on M and $A_{ij}(x) = A^\alpha_{ij}(x)\frac{\partial}{\partial y^\alpha}$ denotes the second fundamental tensor on M. If we set

$$h_{ij}(x) := \langle A_{ij}(x), \nu(x)\rangle\,,$$

then the equation from above gives in view of (5)

$$h_{ij}(x) = \frac{\langle D^2 p(f(x)), f_i(x)\otimes f_j(x)\rangle}{||Dp(f(x))||}\,, \tag{6}$$

where

$$D^2 p(y) := \delta^{\alpha\gamma}\delta^{\beta\delta}\frac{\partial^2 p}{\partial y^\alpha \partial y^\beta}(y)\frac{\partial}{\partial y^\gamma}\otimes\frac{\partial}{\partial y^\delta}$$

is the Hessian of p.

Taking a trace of (6) w.r.t. the induced metric $g_{ij}(x) = \langle f_i(x), f_j(x)\rangle$ yields that the mean curvature $H = g^{ij}h_{ij}$ is given by

$$\begin{aligned}
(7)\,H(x) &= \frac{1}{||Dp(f(x))||}\Big(\Delta_e p(f(x)) - \langle D^2 p(f(x)), \nu(x)\otimes\nu(x)\rangle\Big)\\
&\overset{(5)}{=} \frac{1}{||Dp(f(x))||}\Big(\Delta_e p(f(x)) - \frac{\langle D^2 p(f(x)), Dp(f(x))\otimes Dp(f(x))\rangle}{||Dp(f(x))||^2}\Big)\,.
\end{aligned}$$

Here Δ_e denotes the euclidean Laplace operator in $\mathbb{R}^{n+1}$. Moreover, we have

$$\langle f(x), \nu(x)\rangle = -\frac{\langle Dp(f(x)), f(x)\rangle}{||Dp(f(x))||}\,. \tag{8}$$

In particular, a necessary condition for a selfsimilar solution M to be the the a-level set of a smooth function p is, that the following equation holds at all regular points $y \in M$

$$0 = q[p](y) := ||Dp(y)||^2\Big(\Delta_e p(y) - \mu\langle Dp(y), y\rangle\Big) - \langle D^2 p(y), Dp(y)\otimes Dp(y)\rangle\,. \tag{9}$$

3. Selfsimilar algebraic hypersurfaces

An immersion $f : M \to \mathbb{R}^{n+1}$ is a selfsimilar solution of the mean curvature flow, if there exists a constant $\mu \in \mathbb{R}$ such that

$$H = -\mu\langle f, \nu\rangle\,. \tag{10}$$

The selfsimilar solution is called shrinking, stationary (minimal) or expanding depending on whether $\mu > 0, \mu = 0$ or $\mu < 0$. Here, we are interested in those selfsimilar solutions that are level sets of homogeneous polynomials, hence in homogeneous algebraic solutions. To this end assume p is a homogeneous polynomial of degree d, i.e.

$$p(y) = \sum_{\alpha_1,\dots,\alpha_d=1}^{n+1} c_{\alpha_1\cdots\alpha_d} y^{\alpha_1}\cdots y^{\alpha_d} \tag{11}$$

with a coefficient matrix $c := (c_{\alpha_1\dots\alpha_d})_{\alpha_1,\dots,\alpha_d=1,\dots,n+1}$ being nonzero.

Definition 3.1. A nonempty subset $M_a \subset \mathbb{R}^{n+1}$ is called an algebraic selfsimilar solution of the mean curvature flow of degree d, if there exists a polynomial p of degree d and a constant a such that M_a satisfies

i) $M_a = \{y \in \mathbb{R}^{n+1} : p(y) = a\}$
ii) $q[p](y) = 0\,, \forall y \in M_a$, where $q[p](y)$ is defined as in (9).

Moreover, M_a is called

a) regular w.r.t. p, if a is a regular value of p, i.e. if $Dp(y) \neq 0\,, \forall y \in M_a$,
b) irreducible, if the polynomial $p - a$ is irreducible,
c) homogeneous, if the polynomial p is homogeneous of degree d.

Remark 3.2. *a) From (7) and (8) we obtain that an algebraic selfsimilar solution satisfies*

$$M_a = \left\{y \in \mathbb{R}^{n+1} : p(y) = a\right\} \tag{12}$$

and

$$M_a \subset \left\{y \in \mathbb{R}^{n+1} : q[p](y) = 0\right\} . \tag{13}$$

$q[p]$ is a polynomial of degree at most $3d-2$. Since $p-a$ and $q[p]$ vanish simultaneously on M_a, we conclude that if M_a is irreducible, then there exists a polynomial r of degree at most $2d-2$, such that

$$q[p] = r(p-a)\,. \tag{14}$$

b) For a homogeneous algebraic selfsimilar solution we obtain for any positive constant $\lambda > 0$

$$\lambda \cdot M_a = \{y : y/\lambda \in M_a\} = \{y : p(y/\lambda) = a\} = \{y : p(y) = a\lambda^d\} = M_{a\lambda^d}\,.$$

This shows that if M_a is a homogeneous algebraic selfsimilar solution, then so is $\lambda \cdot M_a$ and the type (shrinking, stationary or expanding) is the same.

c) If p is homogeneous and has degree bigger than 1, then a level set $M_a = \{y \in \mathbb{R}^{n+1} : p(y) = a\}$ can only be regular, if $a \neq 0$. This means that w.l.o.g. we may assume $a \neq 0$, if M_a is a regular homogeneous algebraic selfsimilar solution of degree $d > 1$.

d) Moreover, if p is homogeneous and we write

$$p(y) = \sum_{\alpha_1,\dots,\alpha_d=1}^{n+1} c_{\alpha_1\cdots\alpha_d} y^{\alpha_1}\cdots y^{\alpha_d}$$

then for any permutation $\sigma \in S_d$ we get

$$p(y) = \sum_{\alpha_1,\dots,\alpha_d=1}^{n+1} c_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(d)}} y^{\alpha_{\sigma(1)}}\cdots y^{\alpha_{\sigma(d)}} = \sum_{\alpha_1,\dots,\alpha_d=1}^{n+1} c_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(d)}} y^{\alpha_1}\cdots y^{\alpha_d}$$

as well. This shows that

$$p(y) = \frac{1}{d!}\sum_{\alpha_1,\ldots,\alpha_d=1}^{n+1}\left(\sum_{\sigma\in S_d} c_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(d)}}\right) y^{\alpha_1}\cdots y^{\alpha_d}$$

and w.l.o.g. we may assume that the coefficients $c_{\alpha_1\ldots\alpha_d}$ are fully symmetric.

4. Homogeneous algebraic selfsimilar solutions

Throughout this section we assume that

$$M_a := \{y \in \mathbb{R}^{n+1} : p(y) = a\}$$

is an irreducible homogeneous algebraic selfsimilar solution of the mean curvature flow of degree d. We want to classify all such solutions. To this end let us distinguish between the cases $a \neq 0$ and $a = 0$ (projective varieties).

1) Suppose $a \neq 0$. Since $\langle Dp(y), y\rangle = dp(y) = da \neq 0$ for all $y \in M_a$, we observe that M_a is a regular hypersurface.
   a) Suppose $d = 1$. Let $p(y) = \langle c, y\rangle$ for some nonzero vector $c \in \mathbb{R}^{n+1}$. Then

   $$Dp(y) = c \neq 0\,, \quad D^2p(y) = 0$$

   and

   $$q[p](y) = -\mu||Dp(y)||^2\langle Dp(y), y\rangle = -\mu||c||^2 p(y)\,.$$

   M_a is an affine hyperspace and hence totally geodesic, in particular M_a is minimal and $\mu = 0$, $q[p] = 0$. This, of course, is trivial but we include the computations for the sake of completeness.
   b) Suppose M_a is a quadric, i.e. $d = 2$. Let

   $$p(y) = c_{\alpha\beta}y^\alpha y^\beta\,,$$

   with a symmetric quadratic matrix $(c_{\alpha\beta})_{\alpha,\beta=1,\ldots,n+1} \neq 0$. Then

   $$D_\alpha p(y) = 2c_{\alpha\beta}y^\beta\,, \quad D_\alpha D_\beta p(y) = 2c_{\alpha\beta}\,.$$

   We get

   $$\begin{aligned} q[p](y) &= ||Dp(y)||^2(\Delta_e p(y) - \mu D_\alpha p(y)y^\alpha) - D_\alpha D_\beta p(y)D^\alpha p(y)D^\beta p(y) \\ &= ||2c_{\alpha\beta}y^\beta||^2(2c_\alpha^\alpha - 2\mu c_{\alpha\beta}y^\alpha y^\beta) - 2c_{\alpha\beta}(2c_\gamma^\alpha y^\gamma)(2c_\delta^\beta y^\delta) \\ &= 8||c_{\alpha\beta}y^\beta||^2(c_\alpha^\alpha - \mu p(y)) - 8c_{\alpha\beta}c_\gamma^\alpha c_\delta^\beta y^\gamma y^\delta \\ &= -8\mu||c_{\alpha\beta}y^\beta||^2(p(y) - a) + 8(||c_{\alpha\beta}y^\beta||^2(c_\alpha^\alpha - \mu a) - c_{\alpha\beta}c_\gamma^\alpha c_\delta^\beta y^\gamma y^\delta) \end{aligned}$$

   Consequently, the polynomial

   $$\tilde{q}(y) := d_{\alpha\beta}y^\alpha y^\beta$$

   with

   $$d_{\alpha\beta} := (c_\gamma^\gamma - \mu a)c_\alpha^\delta c_{\delta\beta} - c_{\gamma\delta}c_\alpha^\gamma c_\beta^\delta$$

   must be a multiple of $p(y) - a$. The ansatz

   $$d_{\alpha\beta}y^\alpha y^\beta = k(y)(c_{\alpha\beta}y^\alpha y^\beta - a)$$

   for some polynomial k immediately gives $k = 0$ and then

   $$d_{\alpha\beta} = 0\,.$$

Let $\lambda_1, \dots, \lambda_{n+1}$ be the eigenvalues of the symmetric matrix $c_{\alpha\beta}$. Then $d_{\alpha\beta} = 0$ implies the equations

$$\left(\sum_{i=1}^{n+1} \lambda_i - \mu a\right) \lambda_k^2 - \lambda_k^3 = 0\,, \quad \forall k = 1, \dots, n+1\,.$$

If $\lambda_k = 0$, then this equation is satisfied. Let $\lambda_1, \dots, \lambda_l$ be the nonzero eigenvalues of $c_{\alpha\beta}$. Since $c = (c_{\alpha\beta})_{\alpha,\beta=1,\dots,n+1} \neq 0$ we must have $l \geq 1$. For all nonzero eigenvalues of $c_{\alpha\beta}$ we obtain

$$\left(\sum_{i=1}^{l} \lambda_i - \mu a\right) = \lambda_k\,, \quad k = 1, \dots, l\,.$$

This shows that all nonzero eigenvalues of $c_{\alpha\beta}$ are equal, say $\lambda := \lambda_k$, $k = 1, \dots, l$. We must have $l \geq 2$ since otherwise $l = 1, \mu = 0$ and M_a would be a double plane and thus M_a would be reducable. So we have $l \geq 2$ and $\lambda = \frac{\mu a}{l-1}$. Since $\lambda \neq 0$ we must have $\mu \neq 0$. From $p(y) = c_{\alpha\beta} y^\alpha y^\beta = a$ we obtain that λ and a have the same sign, so $\mu > 0$ and M_a is a self-shrinker. Then $M_a = S^{l-1}(\sqrt{l-1/\mu}) \times \mathbb{R}^{n-l+1}$ with $2 \leq l \leq n+1$.

c) Suppose $d \geq 3$. We have

$$p(y) = c_{\alpha_1 \dots \alpha_d} y^{\alpha_1} \cdots y^{\alpha_d}$$

with a symmetric coefficient matrix $c_{\alpha_1 \dots \alpha_d}$. This gives

$$D_\alpha p(y) = d c_{\alpha\alpha_1 \dots \alpha_{d-1}} y^{\alpha_1} \cdots y^{\alpha_{d-1}}$$

and

$$D_\alpha D_\beta p(y) = d(d-1) c_{\alpha\beta\alpha_1 \dots \alpha_{d-2}} y^{\alpha_1} \cdots y^{\alpha_{d-2}}\,.$$

In particular

$$\langle Dp(y), y \rangle = d p(y)\,. \tag{15}$$

(15) implies

$$\begin{aligned} q[p](y) &= ||Dp(y)||^2 (\Delta_e p(y) - \mu D_\alpha p(y) y^\alpha) - D_\alpha D_\beta p(y) D^\alpha p(y) D^\beta p(y) \\ &= -\mu d ||Dp(y)||^2 (p(y) - a) \\ &\quad + ||Dp(y)||^2 \Delta_e p(y) - D_\alpha D_\beta p(y) D^\alpha p(y) D^\beta p(y) - a\mu ||Dp(y)||^2\,. \end{aligned} \tag{16}$$

Let us set

$$h_{3d-4}(y) := ||Dp(y)||^2 \Delta_e p(y) - D_\alpha D_\beta p(y) D^\alpha p(y) D^\beta p(y)$$

and

$$h_{2d-2}(y) := a\mu ||Dp(y)||^2\,.$$

Note that $d > 2$ implies $3d - 4 > 2d - 2$ so that in particular our notation makes sense. h_{3d-4} and h_{2d-2} are (possibly vanishing) homogeneous polynomials of degree $3d-4$ resp. $2d-2$. Since $p(y) - a$ is irreducible, we know from (16) that there exists a polynomial r of degree at most $2d-4$ with

$$h_{3d-4}(y) - h_{2d-2}(y) = r(y)(p(y) - a)\,.$$

i) $r \neq 0$. The polynomial r can be decomposed into a sum of monomials r_k of degree k

$$r = \sum_{k=0}^{2d-4} r_k \,.$$

Let $k_+, k_- \in \{0, \dots, 2d-4\}$ be the maximal and minimal number k for which $r_k \neq 0$. The maximal and minimal degrees of the monomials in the decomposition for $r(p-a)$ are then $d + k_+$ and k_- since $a \neq 0$ and $p \neq 0$. Since $k_+ \geq k_-$ we have $d + k_+ \neq k_-$. On the other hand $r(p-a) = h_{3d-4} - h_{2d-2}$. This shows that

$$d + k_+ = 3d - 4 \quad \text{and} \quad k_- = 2d - 2 \,.$$

But then $k_+ - k_- = -2$ which is impossible. Hence $r \neq 0$ cannot be true.

ii) Suppose $r = 0$. Then h_{3d-4} and h_{2d-2} both vanish.

$$h_{2d-2}(y) = a\mu||Dp(y)||^2 = 0$$

and $Dp(y) \neq 0$ imply $\mu = 0$. Hence M_a must be a regular minimal algebraic hypersurface.

2) Suppose M_a is a projective variety, i.e. $a = 0$.

a) $d = 1$. This can be treated as in case 1). The difference is, that this time $M_0 := M_a$ is even a linear hyperspace.

b) Suppose $d \geq 2$. M_a is singular because $0 \in M_0 = M_a$ and $Dp(0) = 0$. More precisely, since p is homogeneous, M_0 is a minimal cone Γ with vertex at the origin. The minimality follows from $a = 0$, because then $q[p](y) = 0$ is equivalent to

$$||Dp(y)||^2 \Delta_e p(y) - \langle D^2 p(y), Dp(y) \otimes Dp(y) \rangle = 0$$

which by (7) is equivalent to the vanishing of the mean curvature $H(y)$ (at least at the regular points y of M_a). On the other hand a cone Γ is minimal, if and only if its link $\tilde{\Gamma} := \Gamma \cap S^n$ is minimal as a hypersurface in S^n. Consequently, the classification of $M_a = M_0 = \Gamma$ in this case is equivalent to the classification of all homogeneous algebraic minimal hypersurfaces $\tilde{\Gamma}$ of the sphere of degree $d \geq 2$ (a hypersurface of the sphere is called homogeneous algebraic, if the corresponding cone is the 0-level set of a homogeneous algebraic polynomial). One example is the cone $\Gamma_{\mathbb{T}^2}$ generated by the Clifford torus $\mathbb{T}^2 = \{y \in S^3 : (y^1)^2 + (y^2)^2 = (y^3)^2 + (y^4)^2 = 1/2\}$

$$\Gamma_{\mathbb{T}^2} := \{y \in \mathbb{R}^4 : (y^1)^2 + (y^2)^2 - (y^3)^2 - (y^4)^2 = 0\} \,.$$

If we summarize the results in this section we obtain the proof of Theorem 1.1.

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