MEAN CURVATURE FLOW OF SPACE-LIKE LAGRANGIAN SUBMANIFOLDS IN ALMOST PARA-KÄHLER MANIFOLDS

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Abstract. Given an almost para-Kähler manifold equipped with a metric and para-complex connection, we define a generalized second fundamental form and generalized mean curvature vector of space-like Lagrangian submanifolds. We then show that the deformation induced by this variant of the mean curvature vector field preserves the Lagrangian condition, if the connection satisfies also some Einstein condition. In case the almost para-Kähler structure is integrable, the flow coincides with the classical mean curvature flow in the pseudo-Riemannian context.

1. Introduction

The idea of almost para-complex geometry is to replace the almost complex structure $J \in \mathrm{End}(TN)$ of an almost complex manifold (N, J) subject to the relation $J^2 = -1$ by the almost para-complex structure $P \in \mathrm{End}(TN)$ satisfying $P^2 = 1$. In addition one demands that the eigenbundles $T^{\pm}N := \ker(1 \mp P)$ have the same dimension. A para-complex structure is called integrable if $T^{\pm}N$ define integrable distributions. The obstruction to integrability is given by the analogue of the Nijenhuis-tensor and the Newlander-Nirenberg theorem is a direct consequence of Frobenius' theorem. For more details on para-complex geometry we invite the reader to consult the survey article [CFG] and [AMT] which reviews the developments after the publication of [CFG].

An almost para-hermitian manifold (N, P, h) is a semi-Riemannian manifold (N, h) endowed with a para-complex structure P such that

Date: March 1, 2010.

2000 *Mathematics Subject Classification.* Primary 53C44;

Key words and phrases. Mean curvature flow, almost (para-)Kähler manifolds .

This work was partially written in the framework of the Graduiertenkolleg 1463 "Analysis, Geometry and String Theory". Part of it forms part of the first author's doctoral thesis.

$\omega := h(P\cdot,\cdot)$ defines a two-form, called the characteristic two-form. We observe that this condition implies, that h has split signature, i.e. signature (n,n). An almost para-hermitian manifold is called almost para-Kähler manifold if the characteristic two-form is closed. An almost para-Kähler manifold is para-Kähler if and only if P is parallel w.r.t. the Levi-Civita connection ∇ of h. In this case the bundles $T^{\pm}N$ are integrable and Lagrangian, which motivates the name (almost) bi-Lagrangian manifold for (almost) para-Kähler manifolds.

Our aim in this paper is to study the motion of a Lagrangian submanifold in almost para-Kähler manifolds driven by its mean curvature or a variant of it. It is well known that the mean curvature flow in pseudo-Riemannian manifolds N admits a unique smooth solution for smooth compact initial data M, if one assumes that the initial submanifold M is space-like, i.e. if the pull-back of the pseudo-Riemannian metric h on N to M becomes a Riemannian metric. In such a case the mean curvature vector is an elliptic operator and hence the mean curvature flow becomes parabolic. Therefore the space-like condition is certainly also important in our case of Lagrangian submanifolds in almost para-Kähler manifolds. As we will see later, the classical mean curvature flow will preserve the Lagrangian property only in para-Kähler Einstein manifolds. Since we would also like to formulate a Lagrangian deformation related to the mean curvature vector in the case of Lagrangian submanifolds in almost para-Kähler manifolds it will turn out that we will have to modify the mean curvature and second fundamental form so that (under some Einstein condition) the resulting flow still preserves the Lagrangian property. The construction of the generalized mean curvature and second fundamental form very much depends on a suitable choice of a connection on the almost para-Kähler manifold.

For our purpose these connections lie in the class of *metric* and *para-complex* connections $\widehat{\nabla}$ on M, i.e. connections $\widehat{\nabla}$ in the following set

$$\mathscr{P} := \{\widehat{\nabla} : \widehat{\nabla} h = 0,\ \widehat{\nabla} P = 0\}. \tag{1}$$

Details on $\mathscr{P}$ are discussed in section 2.1. Let $F : M \to (N,P,h)$ be a non-degenerate (i.e. $g := F^*h$ is non-degenerate) immersion and $\widehat{\nabla} \in \mathscr{P}$ then we call $\widehat{\nabla} dF \in \Gamma(F^*TN \otimes T^*M \otimes T^*M)$, given by

$$\hat{A}(X,Y) = \widehat{\nabla}_X(dF)Y = \widehat{\nabla}_{dFX}(dFY) - dF(\nabla^g_X Y), \quad \forall X,Y \in TM$$

the induced second fundamental form.

Definition 1. *A space-like submanifold M in an almost para-Kähler manifold N will be called semi-dimensional, iff* $\dim M = \frac{1}{2}\dim N$.

Taking an appropriate trace of the second fundamental form of a semi-dimensional space-like submanifold we obtain a generalized mean curvature vector $\overrightarrow{\widehat{\mathrm{H}}}$ which differs from the classical mean curvature vector by some lower order terms, where the lower order terms are completely determined by the para-complex structure, since they are computed from the almost para-complex structure P and its Nijenhuis-tensor, cf. Definition 3 for a precise statement. In particular, the lower order terms vanish if P is integrable.

We will consider deformations of space-like Lagrangian submanifolds induced by the generalized mean curvature, i.e. we are interested in families

$$F : M \times [0, T) \to N$$

of space-like Lagrangian immersions in an almost para-Kähler manifold that vary according to the subsequent system of equations

$$\frac{\partial F}{\partial t}(p,t) = \overrightarrow{\widehat{\mathrm{H}}}(p,t), \quad \text{and} \quad F(M,0) = M_0. \tag{2}$$

To justify equation (2) we need to answer two questions:

I.) Does the flow admit a smooth solution in the class of semi-dimensional space-like submanifolds?
II.) Does the flow preserve the Lagrangian condition?

We will see that the first question can be approved in general for compact semi-dimensional space-like submanifolds, more precisely we will prove

Theorem 1. *Let* (N, P, h) *be an almost para-Kähler manifold endowed with a connection* $\widehat{\nabla} \in \mathscr{P}$ *and* M_0 *be a smooth compact semi-dimensional space-like submanifold of* (N, P, h)*. Then there exists a maximal time* $T \in (0, \infty]$ *such that the system (2) admits a smooth solution in the class of semi-dimensional space-like submanifolds on* $[0, T)$ *.*

The second question will only have a positive answer, if we assume in addition that the connection satisfies some Einstein condition. To make this precise let us consider an almost para-Kähler manifold (N, P, h) endowed with a connection $\widehat{\nabla} \in \mathscr{P}$ and let us then define the Ricci*-form $\widehat{\rho}^*$ by

$$\widehat{\rho}^*(V, W) := \frac{1}{2} \operatorname{tr}_h(\widehat{R}(V, W) \circ P).$$

The connection $\widehat{\nabla}$ is called **-Einstein*-connection if its Ricci*-form is a multiple of ω, i.e.

$$\widehat{\rho}^* = f\omega$$

for some smooth function $f \in C^\infty(N)$. Examples of such manifolds are discussed in section 2.2. In general *-Einstein connections are not unique, see Remark 1.

Theorem 2. *Suppose* (N, P, h) *is an almost para-Kähler manifold endowed with a *-Einstein-connection* $\widehat{\nabla}$*. Then the system* (2) *for* $\widehat{\nabla}$ *preserves the Lagrangian condition.*

The generalized mean curvature flow defined in (2) is uniquely defined up to the diffeomorphism group of M. In summary, Theorems 1 and 2 show that there exists a generalized Lagrangian mean curvature flow of space-like Lagrangian submanifolds in almost para-Kähler manifolds admitting a $*$-Einstein connection. Para-Kähler and almost para-Kähler manifolds with such a $*$-Einstein property exist in abundance. For some examples see section 2.2.

Our result can be interpreted as a pseudo-Riemannian analogue of a similar result obtained by M.-T. Wang and the third author [SW, S] and Behrndt [B]. We also mention that mean curvature flow in pseudo-Riemannian manifolds has been studied by a number of people. In [E] and [EH] Ecker and Huisken considered mean curvature flow of space-like hypersurfaces in Lorentzian manifolds. In higher codimension we mention the preprints by Li and Salavessa [LS] resp. by Huang [H]. Minimal and maximal submanifolds in Riemannian and pseudo-Riemannian manifolds have been treated by many authors (cf. [X] for more references).

2. Almost para-Kähler manifolds

In this section we recall some properties of para-complex geometry. In particular, we discuss almost para-Kähler manifolds (or almost bi-Lagrangian manifolds) and explain different classes of examples of such manifolds. For more information we invite the interested reader to consult the survey articles [CFG, AMT, EST].

2.1. Almost para-Kähler manifolds and $*$-Einstein connections.

In this section we consider an *almost para-hermitian structure* (P, h) on a manifold N. This is a tuple of an *almost para-complex structure*, i.e. an endomorphism field $P \in \Gamma(\mathrm{End}(TN))$ which satisfies

$P^2 = 1$ and has eigenbundles $T^{\pm}N := \ker(1 \mp P)$ of the same dimension and a pseudo-Riemannian metric h, such that $h(P\cdot, P\cdot) = -h(\cdot, \cdot)$. This condition forces the metric h to have split signature, i.e. signature (n, n). Equivalently $\omega(\cdot, \cdot) = h(P\cdot, \cdot)$ defines a two-form, called the *fundamental two-form*. An almost *para-hermitian manifold* is a manifold endowed with an almost para-hermitian structure and it is called an *almost para-Kähler* manifold if the fundamental two-form is closed. In particular, almost para-Kähler manifolds are symplectic pseudo-Riemannian manifolds. A para-complex structure is called *integrable* if both $T^{\pm}N$ define integrable distributions. The obstruction to integrability is the Nijenhuis-tensor which is defined as follows:

$$N_P(X, Y) := [X, Y] + [PX, PY] - P[X, PY] - P[PX, Y].$$

Para-hermitian and para-Kähler structures are integrable almost para-hermitian and almost para-Kähler structures. If P is parallel w.r.t. the Levi-Civita connection, then (N, P, h) is a para-Kähler manifold. We note that the classes of almost Kähler manifolds resp. almost para-Kähler manifolds in some sense bear some algebraic similarities but in general they form quite different classes. E.g. $S^{2n} \times S^{2n}$ is always para-Kähler but admits only an almost complex structure in the cases $n = 1, 3$ [DS].

Next we consider an almost para-hermitian manifold (N, P, h). The first *canonical connection* is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2} P(\nabla_X P)Y, \text{ for } X, Y \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection of h. We denote by $\mathscr{P}$ the class of metric and para-complex connections $\widehat{\nabla}$, i.e.

$$\mathscr{P} := \{\widehat{\nabla} : \widehat{\nabla} h = 0,\ \widehat{\nabla} P = 0\}.$$

These properties imply that a connection $\widehat{\nabla} \in \mathscr{P}$ is also *symplectic*, i.e. satisfies the equation $\widehat{\nabla}\omega = 0$ for the fundamental two-form $\omega(V, W) = h(PV, W)$.

Lemma 1. *Let (N, P, h) be an almost para-Kähler manifold:*

1. *Then $\bar{\nabla} \in \mathscr{P}$. In particular, $\mathscr{P}$ is non-empty.*
2. *Let $\widehat{\nabla} \in \mathscr{P}$ and D be a further connection on N. We define*

$$\mathcal{C}(X, Y) := \mathcal{C}_X Y := D_X Y - \widehat{\nabla}_X Y.$$

Then $D \in \mathscr{P}$ if and only if $\mathcal{C}_X$ commutes with P and satisfies $h(\mathcal{C}_X Y, Z) = -h(Y, \mathcal{C}_X Z)$ for $X, Y, Z \in \Gamma(TM)$.

The curvature tensors of D and $\widehat{\nabla}$ are related by

$$R^D(X,Y) = R^{\widehat{\nabla}}(X,Y) + d^{\widehat{\nabla}}\mathcal{C}(X,Y) + [\mathcal{C}_X, \mathcal{C}_Y]$$

and the torsion tensors by

$$T^D(X,Y) = \widehat{T}(X,Y) + \mathcal{C}_X Y - \mathcal{C}_Y X,$$

where $d^{\widehat{\nabla}}$ is the exterior derivative w.r.t. $\widehat{\nabla}$ and $\widehat{T}$ resp. T^D denote the torsions of $\widehat{\nabla}$ resp. of D.

(3) *The difference of a connection $\widehat{\nabla} \in \mathscr{P}$ and the Levi-Civita connection ∇ is given by*

$$\begin{aligned} 2\langle \widehat{\nabla}_X Y - \nabla_X Y, Z\rangle &= \langle \widehat{T}(X,Y), Z\rangle + \langle \widehat{T}(Z,X), Y\rangle \\ &\quad + \langle \widehat{T}(Z,Y), X\rangle, \end{aligned}$$

where $\widehat{T}$ is the torsion of $\widehat{\nabla}$.

(4) *Let (N,P) be an almost para-complex manifold, then there exists a para-complex connection $\widehat{\nabla}$ with torsion $\widehat{T}$ satisfying*

$$\widehat{T}(X,Y) = -\frac{1}{4} N_P(X,Y).$$

If (N,P,h) is almost para-Kähler then this connection coincides with the first canonical connection $\bar{\nabla}$.

Proof. Part (1) follows as $P\nabla_X P$ is anti-symmetric w.r.t. h and anti-commutes with P. Part (2) is a direct computation. Part (3) can be shown by the same procedure as the Koszul-formula by taking care of the torsion $\widehat{T}$. The connection in part (4) is given by $\widehat{\nabla} = \nabla + Q$, where Q is defined as follows

$$4Q(X,Y) := [(\nabla_{PY} P)X + P((\nabla_Y P)X) + 2P((\nabla_X P)Y)].$$

It is well-known, that two metric connections with the same torsion coincide. An almost para-Kähler manifold is quasi-para-Kähler, i.e. it satisfies

$$(\nabla_{PY} P)X = -P(\nabla_Y P)X$$

and in consequence it follows $4Q = 2P((\nabla_X P)Y)$ and $\widehat{\nabla} = \bar{\nabla}$. □

A special situation is the deformation of the connection in direction of the para-complex structure in the sense of the next corollary.

Corollary 1. *Let $\widehat{\nabla} \in \mathscr{P}$ and $\sigma \in \Omega^1(N)$ be arbitrary and set $D := \widehat{\nabla} + \sigma \otimes P \in \mathscr{P}$. Then the curvature tensors R^D and $\widehat{R}$ of D resp. $\widehat{\nabla}$ are related by*

$$R^D = \widehat{R} + d\sigma \otimes P$$

and the torsions T^D and $\widehat{T}$ are related by

$$T^D = \widehat{T} + \sigma \wedge P\,.$$

Lemma 2. *Let (N, P, h) be an almost para-Kähler manifold. Then the curvature tensor $\widehat{R}$ of a connection $\widehat{\nabla} \in \mathscr{P}$ has the symmetries*

$$\begin{aligned} h(\widehat{R}(X,Y)V,W) &= -h(\widehat{R}(Y,X)V,W) = -h(\widehat{R}(X,Y)W,V)\,, && (3)\\ h(\widehat{R}(X,Y)PV,W) &= h(P\widehat{R}(X,Y)V,W) = h(\widehat{R}(X,Y)PW,V) && (4)\end{aligned}$$

and

$$\omega(\widehat{R}(V,W)X,Y) = \omega(\widehat{R}(V,W)Y,X)\,. \qquad (5)$$

In particular $h(\widehat{R}(X,Y)P\cdot,\cdot) = \omega(\widehat{R}(X,Y)\cdot,\cdot)$ is a symmetric tensor.

Proof. The first identity follows, since the connection is metric. The second, since it is para-complex. The third symmetry follows, as $\widehat{\nabla}$ is metric and para-complex and in consequence symplectic. □

Since $h(\widehat{R}(X,Y)P\cdot,\cdot)$ is symmetric one can define the Ricci*-form $\widehat{\rho}^*$ by

$$\widehat{\rho}^*(X,Y) := \frac{1}{2}\,\mathrm{tr}_h(\widehat{R}(X,Y)\circ P)\,.$$

The Ricci*-form $\widehat{\rho}^*$ w.r.t. ω is given by

$$\widehat{\rho}^*(X,Y) := \frac{1}{2}\sum_{\alpha=1}^{2n}\omega(\widehat{R}(X,Y)e_\alpha, e_\alpha)\,,$$

where e_α is an arbitrary orthonormal basis of TN.

Definition 2. *Let (N, P, h) be an almost para-Kähler manifold. A connection $\widehat{\nabla} \in \mathscr{P}$ is called *-Einstein, if it satisfies $\widehat{\rho}^* = f\omega$ for some function f on N.*

We will see that almost para-Kähler manifolds with *-Einstein connections form the suitable class of almost para-Kähler manifolds in the sense that the generalized mean curvature flow defined in (2) preserves the Lagrangian condition.

2.2. Examples of almost para-hermitian manifolds.

In this section we will give several examples for almost para-Kähler manifolds.

2.2.1. *Products.*

Let (M, g) be a Riemannian manifold and consider the product $N = M \times M$ endowed with the para-complex structure given by

$$P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the metric $h = \pi_1^* g \oplus (-\pi_2^* g)$ with respect to the decomposition

$$T_{(m_1,m_2)}N = T_{m_1}M \oplus T_{m_2}M,$$

where π_i, with $i = 1, 2$, denotes the projection onto the factors. This structure P is parallel for the Levi-Civita connection of N which preserves the decomposition of TN. In particular, this yields a class of para-Kähler manifolds. Moreover, the curvature tensor preserves the decomposition of TN. Hence these examples are *-Einstein with respect to the connection $\bar{\nabla} = \nabla$ and the Ricci*-form vanishes.

2.2.2. *Cotangent bundles as almost para-Kähler manifolds.*

In the following class of examples we follow the notations in [V] and [YI]. Let (M^n, g) be a Riemannian manifold and $\pi : N^{2n} = T^*M \to M$ its cotangent bundle. We consider the Levi-Civita connection ∇^g on TM and decompose

$$TN = \mathcal{H} \oplus \mathcal{V},$$

where the vertical part $\mathcal{V}$ is defined as $\mathcal{V} := \ker d\pi$ and the horizontal part $\mathcal{H}$ is induced by the parallel transport with respect to ∇^g.

Let us take the local coordinates x^i, ξ_i on T^*M. On overlapping charts with coordinates x^i, ξ_i and $\tilde{x}^i, \tilde{\xi}_i$, the transformation rule

$$\tilde{\xi}_i = \frac{\partial x^j}{\partial \tilde{x}_i} \xi_j$$

is satisfied. Denote the canonical symplectic form by $\omega = \sum_{i=1}^n dx^i \wedge d\xi_i$ and the Liouville form by $\lambda = \xi_i dx^i$. The vertical distribution is generated by $\frac{\partial}{\partial \xi_i}$ and the horizontal distribution by

$$Y^i = g^{ij} X_j = g^{ij} \frac{\partial}{\partial x^j} + g^{ij} (\theta_j)^k_h \xi_k \frac{\partial}{\partial \xi_h}, \quad i = 1, \ldots, n, \tag{6}$$

where θ is the connection form of the Levi-Civita connection of the metric g. The connection form $\hat{\theta}$ of $\widehat{\nabla}$ in the frame Y^i, $\frac{\partial}{\partial \xi_i}$ is given by

$$\hat{\theta} := - \begin{pmatrix} \theta^i_j & 0 \\ 0 & \theta^i_j \end{pmatrix}. \tag{7}$$

This connection preserves the horizontal and the vertical distribution. Moreover, Y^i and $\frac{\partial}{\partial \xi_i}$ are parallel in the fiber direction, as $\hat{\theta}$ does not depend on these directions. By equation (7) the vector bundle map $\tau : \mathcal{H} \to \mathcal{V}$ sending Y^i to $\frac{\partial}{\partial \xi_i}$ is parallel with respect to $\widehat{\nabla}$ and we define the candidate for the para-complex structure by

$$P := \begin{pmatrix} 0 & \tau \\ \tau^{-1} & 0 \end{pmatrix}$$

and for the metric h by

$$h := \begin{pmatrix} g^{ij} & 0 \\ 0 & -g^{ij} \end{pmatrix}.$$

With these definitions the fundamental two-form $h(P\cdot,\cdot)$ coincides with the canonical symplectic form ω of $N = T^*M$. Since $\hat{\theta}$ is skew-symmetric it follows $\widehat{\nabla} h = 0$. The curvature form $\hat{\Omega}$ of $\widehat{\nabla}$ is computed as

$$\hat{\Omega} = -\begin{pmatrix} \Omega^i_j & 0 \\ 0 & \Omega^i_j \end{pmatrix} \quad \text{and} \quad \hat{\Omega} \circ P = -\begin{pmatrix} 0 & \Omega^i_j \\ \Omega^i_j & 0 \end{pmatrix}, \tag{8}$$

where Ω denotes the curvature form of the Levi-Civita connection ∇^g.

Taking the trace with respect to h yields $\hat{\rho}^* = 0$. This gives a second class of *-Einstein almost para-Kähler manifolds with vanishing Ricci*-form.

We recall, that the torsion tensor $\hat{T}$ of $\widehat{\nabla}$ and the curvature tensor of (M, g) are related by the formula

$$\hat{T}(\cdot,\cdot) = \sum_{ij} \Omega^i_j \xi_i \frac{\partial}{\partial \xi_j}. \tag{9}$$

Therefore one has $\hat{T} = 0$ if and only if (M, g) is flat. In consequence $(N = T^*M, P, h)$ is not para-Kähler unless (M, g) is flat.

Remark 1. *We observe, that the *-Einstein connections are not unique. In fact, any connection* $\tilde{\nabla} := \widehat{\nabla} + c\lambda \otimes P$*, where* λ *is the Liouville form of* T^*M *and* c *some constant is also *-Einstein by Corollary 1.*

2.2.3. *Almost para-hermitian symmetric spaces.*

The class of almost para-Kähler $*$-Einstein manifolds generalizes para-Kähler Einstein manifolds (M, P, h). In fact, if (M, P, h) is para-Kähler Einstein then the Levi-Civita connection ∇^h is a connection in $\mathscr{P}$ and the difference between the Ricci* tensor and the Ricci tensor for the Levi-Civita connection can be expressed in terms of the Nijenhuis tensor (cf. [IZ]) and since the Nijenhuis tensor vanishes for a para-Kähler manifold, para-Kähler Einstein manifolds are also almost para-Kähler $*$-Einstein.

An example of a para-Kähler Einstein manifold is obviously given by $\mathbb{R}^n \times \mathbb{R}^n$ with its para-complex structure discussed in section 2.2.1. A second example is the para-complex projective space CP^n which is of constant para-holomorphic sectional curvature [GMA], i.e. the analogue of the projective space $\mathbb{C}P^n$.

More generally we might consider para-Kähler manifolds with homogeneous para-complex structures. Examples are para-hermitian symmetric spaces which were introduced in [K, KK]. In the same references an infinitesimal classification of para-hermitian symmetric spaces was obtained. A classification of homogeneous para-Kähler Einstein manifolds is given in Theorem 1.1 of [AMT]. In fact, it is shown that for a homogeneous manifold G/H (where G is a semi-simple Lie group) endowed with an invariant para-Kähler structure P there exists a unique invariant symplectic structure ω such that $h_{\lambda,P} := \lambda^{-1}\omega \circ P$ is an invariant para-Kähler Einstein metric with Einstein constant $\lambda \neq 0$. In particular this result provides interesting examples.

2.3. **Submanifolds in almost para-Kähler manifolds.**

In this subsection we consider a smooth space-like immersion $F : M \to (N, P, h)$ into an almost para-Kähler manifold endowed with a fixed connection $\widehat{\nabla} \in \mathscr{P}$. There exist two naturally associated second fundamental forms given by

$$\widehat{A}(X,Y) := \widehat{\nabla}_{dFX}(dFY) - dF(\nabla_X Y) =: \widehat{\nabla}_X(dF)Y \,, \forall X, Y \in \Gamma(TM)$$

and

$$A(X,Y) := \nabla^h_{dFX}(dFY) - dF(\nabla^g_X Y) =: \nabla_X(dF)Y \,, \forall X, Y \in \Gamma(TM),$$

where ∇^h, ∇^g denote the Levi-Civita connections of h resp. $g = F^*h$. In the sequel and by some abuse of notion we will omit the indices in ∇^g and ∇^h and we will simultaneously use the symbols $\widehat{\nabla}, \nabla$ for all connections on bundles over M that are induced from (∇^h, ∇^g) resp. $(\widehat{\nabla}, \nabla^g)$.

From the observation, that the second term in the definition of A and $\widehat{A}$ is the same, one derives the subsequent Lemma.

Lemma 3. *Let $F : M \to (N, P, h)$ be a smooth space-like immersion into an almost para-Kähler manifold (N, P, h) endowed with a fixed* *connection* $\widehat{\nabla} \in \mathscr{P}$. *Then we have*

$$\widehat{A}(X,Y) - A(X,Y) = F^*\mathcal{C}(X,Y)$$

with $\mathcal{C}$ *as defined in Lemma 1 and*

$$\widehat{A}(X,Y) - \widehat{A}(Y,X) = F^*\widehat{T}(X,Y),$$

where $X, Y \in \Gamma(TM)$.

Let us consider the differential dF as a section in the vector bundle $E = F^*TN \otimes T^*M$. Denote by $\widehat{\nabla}$ the product connection on E induced by $\widehat{\nabla}$ (on F^*TN) and the Levi-Civita connection ∇^g (on T^*M) and by ∇ the connection on E induced by the Levi-Civita connections ∇^h (on F^*TN) and ∇^g (on T^*M).

Then we obtain

$$\widehat{\nabla}^2_{X,Y}\sigma - \widehat{\nabla}^2_{Y,X}\sigma = \widehat{R}(dFX, dFY)\sigma - \sigma(R(X,Y)\cdot), \quad \sigma \in \Gamma(E),$$

where we used that the affine connection on TM is torsion-free. Using the definition of $\hat{A} = \widehat{\nabla}(dF)$ and setting $\sigma = dF$ then yields the Codazzi equation:

$$\begin{aligned}(\widehat{\nabla}_X\hat{A})(Y,Z) \;\; - \;\; (\widehat{\nabla}_Y\hat{A})(X,Z) &= (\widehat{\nabla}^2_{X,Y}dF)Z - (\widehat{\nabla}^2_{Y,X}dF)Z \quad (10)\\ &= \widehat{R}(dFX, dFY)dFZ - dF(R(X,Y)Z).\end{aligned}$$

2.4. **Semi-dimensional space-like submanifolds in almost para-Kähler manifolds.** In this subsection we consider semi-dimensional space-like submanifolds. These admit a canonical isomorphism of TM and $T^\perp M$ as is shown in the next lemma.

Lemma 4. *Let* $F : M \to (N, P, h)$ *be a space-like immersion into an almost para-Kähler manifold with* $\dim M = n = \frac{1}{2}\dim N$. *Then the map*

$$\phi : TM \to T^\perp M\,, \quad \phi V := (P(F_*V))^\perp$$

is an isomorphism and

$$T_{(F(p))}N = F_*(T_pM) \oplus T_pM^\perp\,, \quad \forall p \in M\,.$$

Proof. Let $W \in T_pM$ be arbitrary, $W \neq 0$, $V := F_*W$. Then

$$\begin{aligned} h(\phi W, \phi W) &= h(PV - (PV)^\top, PV - (PV)^\top)\\ &= -h(V,V) - 2h(PV, (PV)^\top) + h((PV)^\top, (PV)^\top)\\ &= -h(V,V) - h((PV)^\top, (PV)^\top) < 0\,.\end{aligned}$$

Since $\dim M = \frac{1}{2}\dim N$ this implies ϕ is an isomorphism. Since h is positive definite on $F_*(T_pM)$ and negative definite on $T_pM^\perp$ we obtain for dimensional reasons that $T_{(F(p))}N = F_*(T_pM) \oplus T_pM^\perp$. □

We observe, that due to Lemma 4 semi-dimensional space-like submanifolds are automatically almost Lagrangian in the sense of [SW].

In the rest of this section we fix an almost para-Kähler manifold (N, P, h) endowed with a connection $\widehat{\nabla} \in \mathscr{P}$ and we consider a space-like[1] semi-dimensional immersion $F : M \to N$.

In a basis F_i of $(F_*TM)_p \subset (F^*TN)_p$, $p \in M$, the map ϕ_p is given by

$$\begin{aligned}\phi(V) &= P(F_*V) - \sum_{i,j=1}^{n} h(P(F_*V), F_i)_p g^{ij} F_j \\ &= P(F_*V) - \sum_{i,j=1}^{n} \omega(F_*V, F_i) g^{ij} F_j,\end{aligned}$$

where $V \in T_pM$ and g^{ij} is the inverse of the matrix $g_{ij} = h(F_i, F_j)$.

Inspired by Lagrangian submanifolds we consider for semi-dimensional space-like submanifolds the tensor-fields

$$\widehat{h}(X, Y, Z) := h(\phi(F_* X), \widehat{A}(Y, Z))\,, \text{ for } X, Y, Z \in \Gamma(TM) \tag{11}$$

and

$$\widehat{H} := \mathrm{tr}_g^{1,3}\widehat{h}\,.$$

We call $\widehat{H}$ the generalized mean curvature form of the semi-dimensional space-like submanifold $F : M \to N$. The tensor-field $\widehat{h}(\cdot,\cdot,\cdot)$ decomposes as follows w.r.t. the components in F_*TM and $T^\perp M$

$$\widehat{h}(X, Y, Z) = \widehat{r}(X, Y, Z) - \omega(F_*X, \widehat{s}(Y, Z))\,, \tag{12}$$

$$\widehat{H} = \mathrm{tr}_g^{1,3}\widehat{r} + \mathrm{tr}_{F^*\omega}^{1,3}\widehat{s}\,, \tag{13}$$

where the symbols $\widehat{r}$ and $\widehat{s}$ are simultaneously used for the tensors defined by

$$\widehat{r}(Y, Z) := \widehat{A}(Y, Z)^\perp \text{ and } \widehat{r}(X, Y, Z) := \omega(F_* X, \widehat{A}(Y, Z))\,, \tag{14}$$

$$\widehat{s}(Y, Z) := \widehat{A}(Y, Z)^\top \text{ and } \widehat{s}(X, Y, Z) := h(F_*X, \widehat{A}(Y, Z)). \tag{15}$$

Here and in the following we denote by $\mathrm{tr}_b^{k,l}T$ the trace/contraction of the slots k and l of a tensor field T with respect to a bi-linear form b.

[1] i.e. the induced metric $g := F^*h$ is positive definite

Lemma 5. *With the above definitions one has*

$$\widehat{s}(X,Y,Z) = -\widehat{s}(Z,Y,X)\,. \tag{16}$$

The covariant differential of $F^*\omega$ *and* $\widehat{r}$ *are related by*

$$\begin{aligned}(\nabla_X F^*\omega)(Y,Z) &= \omega(\widehat{A}(X,Y),F_*Z) + \omega(F_*Y,\widehat{A}(X,Z)) \\ &= \widehat{r}(Y,X,Z) - \widehat{r}(X,Y,Z)\,,\end{aligned} \tag{17}$$

where ∇ *denotes the Levi-Civita connection on* M*.*

Proof. The identity (16) follows as $\widehat{\nabla}$ is metric. Since $\widehat{\nabla}$ is symplectic, we get (17) □

Lemma 6. *Let* (N,P,h) *be an almost para-Kähler manifold endowed with a connection* $\widehat{\nabla}\in\mathscr{P}$ *and* $F:M\to N$ *be a smooth semi-dimensional space-like immersion, then one has*

$$d\widehat{H} = (C \lrcorner F^*\omega) - F^*\widehat{\rho}^* \tag{18}$$

for some smooth tensor field C*. In particular, if* F *is Lagrangian and* $\widehat{\nabla}$ *-Einstein, then* $\widehat{H}$ *is closed.*

Here and in the following any contractions of $F^*\omega$ with some tensor C using the metric tensor g will be denoted by an expression of the form $C \lrcorner F^*\omega$.

Proof. We compute the exterior derivative $d\widehat{H}$ via

$$d\widehat{H}(X,Y) = (\nabla_X\widehat{H})Y - (\nabla_Y\widehat{H})X,$$

where ∇ denotes the Levi-Civita connection of M. The expression for $\nabla\widehat{H}$ decomposes by the identity (12) into two parts. First we consider the term $\mathrm{tr}_g^{2,4}(\nabla\widehat{r})$ with $\hat{r}(\cdot,\cdot,\cdot) = \omega(dF\cdot,\hat{A}(\cdot,\cdot))$ and derive

$$\begin{aligned}(\nabla_X\widehat{r})(Y,Z,W) &= \overbrace{X\omega(F_*Y,\hat{A}(Z,W))}^{\omega(\widehat{\nabla}_X(F_*Y),\hat{A}(X,Z))+\omega(F_*Y,\widehat{\nabla}_X(\hat{A}(Z,W)))} \\ &- \widehat{r}(\nabla_XY,Z,W) - \widehat{r}(Y,\nabla_XZ,W) - \widehat{r}(Y,Z,\nabla_XW) \\ &= \omega(\hat{A}(X,Y),\hat{A}(Z,W)) + \omega(F_*Y,(\widehat{\nabla}_X\hat{A})(Z,W)).\end{aligned}$$

Setting $Y=W$ the first term is anti-symmetric in X and Z. The anti-symmetrization of the second term simplifies by Codazzi's equation to

$$\omega(F_*Y,\hat{R}(F_*X,F_*Z)F_*Y) - \omega(F_*Y,F_*R(X,Z)Y).$$

To analyze the first term we fix a local frame F_m of F^*TN and obtain

$$\begin{aligned}\widehat{A}(X,Y) &= h(\widehat{A}(X,Y),F_m)g^{mn}F_n + h(\widehat{A}(X,Y),\phi F_m)\eta^{mn}\phi F_n \\ &= \widehat{s}(F_m,X,Y)g^{mn}F_n + \eta^{mn}(\widehat{r}(F_m,X,Y) \\ &\quad -\omega(F_m,\widehat{s}(X,Y)))\phi F_n,\end{aligned}$$

where η^{mn} is the inverse of the matrix $\eta_{mn} = h(\phi F_n, \phi F_m)$ and g^{mn} is the inverse of the matrix $g_{mn} = h(F_n, F_m)$. In the sequel raised indices will always be raised with respect to the metric tensor g^{ij}. The only exception is η^{ij} which indicates the inverse of η_{ij}. Moreover we set $\omega_{ij} := \omega(F_i, F_j)$ etc.

Further we have the relations

$$\omega(F_m,\phi F_n) = \eta_{mn}\,, \quad \omega(\phi F_p,\phi F_q) = -\omega_{pq} + \omega_p{}^k\omega_q{}^l\omega_{kl}, \tag{19}$$

which imply after a computation

$$\begin{aligned}2\mathrm{tr}_g\omega(\hat{A}(X,\cdot),\hat{A}(Z,\cdot)) &= (C_1\lrcorner F^*\omega)(X,Z) + 2\langle\widehat{r}(\cdot,Z,\cdot),\widehat{s}(\cdot,X,\cdot)\rangle \\ &\quad -2\langle\widehat{r}(\cdot,X,\cdot),\widehat{s}(\cdot,Z,\cdot)\rangle\,. \end{aligned}\tag{20}$$

The second term of $\nabla_X\widehat{H}$ is $\nabla_X(\mathrm{tr}^{1,3}_{F^*\omega}\widehat{s})$ with $\hat{s}(\cdot,\cdot,\cdot) = h(dF\cdot,\hat{A}(\cdot,\cdot))$ and decomposes into two summands:

$$\nabla_X(\mathrm{tr}^{1,3}_{F^*\omega}\widehat{s}) = \mathrm{tr}^{1,3}_{F^*\omega}(\nabla_X\widehat{s}) + \mathrm{tr}^{1,3}_{\nabla_X(F^*\omega)}(\widehat{s}).$$

The first term of this expression is seen to be of the form $C_2\lrcorner F^*\omega$. Using the symmetry (16) of $\widehat{s}$ and the relation of $\widehat{r}$ and $\nabla(F^*\omega)$ (cf. equation (17)) we obtain for the second term

$$\mathrm{tr}^{1,3}_{\nabla_X(F^*\omega)}(\widehat{s}) = 2\mathrm{tr}^{1,3}_{\widehat{r}(\cdot,X,\cdot)}(\widehat{s}). \tag{21}$$

Summarizing we get

$$\begin{aligned}d\hat{H}(X,Y) &= \mathrm{tr}^{2,4}_g(\nabla\widehat{r})(X,Y) - \mathrm{tr}^{2,4}_g(\nabla\widehat{r})(Y,X) \\ &\quad - \nabla(\mathrm{tr}^{1,3}_{F^*\omega}\widehat{s})(X,Y) + \nabla(\mathrm{tr}^{1,3}_{F^*\omega}\widehat{s})(Y,X) \\ &= 2\mathrm{tr}^{2,4}_g\left[\omega(\hat{A}(\cdot,\cdot),\hat{A}(\cdot,\cdot))\right](X,Y) \\ &\quad + \mathrm{tr}^{2,4}_g\left[\omega(F_*\cdot,\hat{R}(F_*\cdot,F_*\cdot)F_*\cdot) - \omega(F_*\cdot,F_*R(\cdot,\cdot)\cdot)\right](X,Y) \\ &\quad + (\mathrm{tr}^{1,3}_{F^*\omega}\nabla_X\widehat{s})(Y) + \mathrm{tr}^{1,3}_{\nabla_X(F^*\omega)}(\widehat{s})(Y) \\ &\quad - (\mathrm{tr}^{1,3}_{F^*\omega}\nabla_Y\widehat{s})(X) - \mathrm{tr}^{1,3}_{\nabla_Y(F^*\omega)}(\widehat{s})(X) \\ &= \mathrm{tr}^{2,4}_g\left[\omega(F_*\cdot,\hat{R}(F_*\cdot,F_*\cdot)F_*\cdot)\right](X,Y) \\ &\quad - \mathrm{tr}^{2,4}_g\left[\omega(F_*\cdot,\hat{R}(F_*\cdot,F_*\cdot)F_*\cdot)\right](Y,X) + (C_3\lrcorner F^*\omega)(X,Y),\end{aligned}$$

where we emphasize the relative sign in equation (20) and (21). We are next going to relate the curvature part of the last expression to the Ricci*-form. Evaluating the Ricci*-form using a local frame F_i of F^*TN yields

$$\widehat{\rho}^*(V,W) = \frac{1}{2}g^{ij}\omega(\widehat{R}(V,W)F_i,F_j) + \frac{1}{2}\eta^{ij}\omega(\widehat{R}(V,W)\phi F_i,\phi F_j)\,.$$

The summands of the second term simplify as follows

$$\begin{aligned}\omega(\widehat{R}(V,W)\phi F_i,\phi F_j) &= \omega(\widehat{R}(V,W)(PF_i-(PF_i)^\top),PF_j-(PF_j)^\top)\\ &= \omega(P\widehat{R}(V,W)F_i,PF_j)+(C_4\lrcorner F^*\omega)(V,W)\\ &= -\omega(\widehat{R}(V,W)F_i,F_j)+(C_4\lrcorner F^*\omega)(V,W).\end{aligned}$$

Using $g^{ij}=g^{im}\delta_m{}^j$ and $\delta_m{}^j=\eta_{mk}\eta^{kj}$ we observe

$$\eta^{ij} = -g^{ij} - \omega^{il}\omega_{sl}\eta^{sj}\,,$$

which yields

$$\widehat{\rho}^*(V,W) = g^{ij}\omega(\widehat{R}(V,W)F_i,F_j) + (C_5\lrcorner F^*\omega)(V,W) \tag{22}$$

and finishes the proof of the Lemma. □

3. Mean curvature flow of Lagrangian submanifolds in almost para-Kähler manifolds

3.1. **The generalized mean curvature vector.**

In this section we fix a semi-dimensional space-like submanifold $F: M\to N$ of an almost para-Kähler manifold $(N,h,P,\omega:=h(P\cdot,\cdot))$. The first variation of the volume induced by the Riemannian metric $g=F^*h$ gives rise to the classical definition of the mean curvature vector field $\overrightarrow{\mathrm{H}}$ on M given by

$$\overrightarrow{\mathrm{H}} = (\mathrm{tr}_g A)^\perp.$$

Definition 3. *Let $(N,h,P,\omega:=h(P\cdot,\cdot))$ be an almost para-Kähler manifold and $F: M\to N$ be a semi-dimensional space-like submanifold. The generalized mean curvature vector field is defined as*

$$\overrightarrow{\widehat{\mathrm{H}}} \;:=\; \overrightarrow{\mathrm{H}} - tr_g(\widehat{T}^t + (\phi^t)^{-1}\widehat{T}\phi),$$

where $\widehat{T}_X$ denotes the operator constructed from the torsion tensor $\widehat{T}$ by $h(\widehat{T}(X,Y),Z)=h(\widehat{T}_XY,Z)$ and $\cdot^t$ is the transposition w.r.t. the metric h.

The next lemma relates the mean curvature vector $\overrightarrow{\mathrm{H}}$ to the mean curvature form $\widehat{H}$.

Lemma 7. *The generalized mean curvature vector $\overrightarrow{\widehat{\mathrm{H}}}$ and the generalized mean curvature form $\widehat{H}$ with respect to the connection $\widehat{\nabla}$ are related by*

$$\begin{aligned} h(\overrightarrow{\widehat{\mathrm{H}}}, \phi\cdot) &= \widehat{H} - tr_g^{1,2}\nabla(F^*\omega) - tr_g^{2,3}\omega(F_*\cdot, \widehat{A}(\cdot,\cdot)) \\ &+ tr_g^{1,2}\omega(F_*\cdot, \widehat{A}(\cdot,\cdot))\,, \end{aligned} \tag{23}$$

In particular, if M is Lagrangian, then

$$\overrightarrow{\widehat{\mathrm{H}}} = g^{ij}\widehat{H}_i PF_j\,. \tag{24}$$

We observe that $d^\dagger(F^*\omega) = \mathrm{tr}_g^{1,2}\nabla(F^*\omega)$ is the co-differential of $F^*\omega$.

Proof. From the definition of $\widehat{h}$ in equation (11) and Lemma 5 we conclude

$$\begin{aligned} \widehat{h}(X,Y,Z) &= \widehat{h}(Z,Y,X) + \nabla_Y(F^*\omega)(X,Z) \\ &\quad -\omega(F_*X, \widehat{s}(Y,Z)) + \omega(F_*Z, \widehat{s}(Y,X)). \end{aligned} \tag{25}$$

Moreover, since $\widehat{\nabla}$ has torsion we get

$$\widehat{h}(X,Y,Z) = \widehat{h}(X,Z,Y) + h(\widehat{T}(F_*Y, F_*Z), \phi F_*X)\,. \tag{26}$$

From these equations we then deduce

$$\begin{aligned} \mathrm{tr}_g^{2,3}\widehat{h} + \mathrm{tr}_g^{2,3}h(\widehat{T}(\cdot,\cdot),\phi\cdot) &= \widehat{H} - \mathrm{tr}_g^{1,2}(\nabla F^*\omega) \\ &\quad -\mathrm{tr}_g^{2,3}\omega(F_*\cdot, \widehat{s}(\cdot,\cdot)) + \mathrm{tr}_g^{1,2}\omega(F_*\cdot, \widehat{s}(\cdot,\cdot)). \end{aligned} \tag{27}$$

In addition, Lemma 1 part (3) implies

$$\mathrm{tr}_g^{2,3}\widehat{h} = h(\mathrm{tr}_g\widehat{A}, \phi\cdot) = h(\overrightarrow{H}, \phi\cdot) + \mathrm{tr}_g^{2,3}h(\widehat{T}(\phi\cdot,\cdot),\cdot)\,.$$

Combining this with (27) we finally get

$$\begin{aligned} h(\overrightarrow{\widehat{\mathrm{H}}}, \phi\cdot) &= h(\overrightarrow{H}, \phi\cdot) + \mathrm{tr}_g^{2,3}h(\widehat{T}(\phi\cdot,\cdot),\cdot) + \mathrm{tr}_g^{2,3}h(\widehat{T}(\cdot,\cdot),\phi\cdot) \\ &= \widehat{H} - \mathrm{tr}_g^{1,2}\nabla(F^*\omega) - \mathrm{tr}_g^{2,3}\omega(F_*\cdot, \widehat{s}(\cdot,\cdot)) + \mathrm{tr}_g^{1,2}\omega(F_*\cdot, \widehat{s}(\cdot,\cdot))\,. \end{aligned}$$

This proves the lemma. □

3.2. **Proof of main theorems.**

We now prove **Theorem 1**:

Proof. It suffices to prove that the operator $E[F] := \overrightarrow{\widehat{\mathrm{H}}}[F]$ has no non-trivial degeneracies, i.e. we have to show that it is elliptic in the normal directions. By Definition 3, the generalized mean curvature vector differs from the classical mean curvature vector only by terms of lower order so that the symbol of our operator is the same as for the mean curvature flow and short-time existence follows. □

Next we prove **Theorem 2**:

Proof. From Cartan's formula we know

$$\frac{\partial}{\partial t} F^*\omega = d\left(F^*\left(\frac{\partial F}{\partial t} \lrcorner \omega\right)\right) = d(F^*(\overrightarrow{\widehat{\mathrm{H}}} \lrcorner \omega)) \,.$$

We fix some time interval $[0, t_0]$, $0 < t_0 < T$. From (18), the *-Einstein property of $\widehat{\nabla}$, and (24) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} F^*\omega &= -d\widehat{H} + dd^\dagger(F^*\omega) + C_1 \lrcorner F^*\omega + C_2 \lrcorner \nabla(F^*\omega) \\ &= dd^\dagger(F^*\omega) + C_3 \lrcorner F^*\omega + C_2 \lrcorner \nabla(F^*\omega) \end{aligned}$$

for smooth tensor fields C_1, C_2, C_3. Since ω (and $F^*\omega$) is closed we have

$$\Delta(F^*\omega) = dd^\dagger(F^*\omega) + C_4 \lrcorner F^*\omega \,,$$

where C_4 depends on the Riemannian curvature of M. Combining the last identities we deduce

$$\frac{\partial}{\partial t} F^*\omega = \Delta(F^*\omega) + C_5 \lrcorner F^*\omega + C_2 \lrcorner \nabla(F^*\omega)$$

for smooth tensor fields C_2, C_5.

Together with Cauchy-Schwarz' inequality, we thus obtain an estimate of the form

$$\frac{\partial}{\partial t}|F^*\omega|^2 \leq \Delta|F^*\omega|^2 + c|F^*\omega|^2$$

for all $t \in [0, t_0]$ and some constant c depending on t_0.

Thus the growth rate of $|F^*\omega|^2$ on $[0, t_0]$ is at most exponential, i.e.

$$\sup_{p\in M} |F^*\omega|^2(p,t) \leq \sup_{p\in M} |F^*\omega|^2(p,0)e^{ct} \,, \forall t \in [0, t_0].$$

However, $|F^*\omega|^2(p,0)$ is zero for all $p \in M$ as M_0 is Lagrangian. Since t_0 is arbitrary, the theorem follows. □

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